DIPLOMARBEIT

Monoidale 2-Funktoren und Spaltensteinvarietäten
(Monoidal 2-functors and Spaltenstein varieties)

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Introduction

English introduction

This thesis consists of two parts, rather different in nature. The first deals with different 2-categories of cobordisms and generalisations of two-dimensional topological quantum field theories (TQFTs). The second one is algebro-geometric and considers very special examples of Springer fibres and Spaltenstein varieties. The two parts will then be connected at the end of the thesis in the main theorem (Theorem 8.5) and Conjecture 8.6.

Introduction to Part I

In this part, we consider and describe in detail three special examples of semistrict monoidal 2-categories and monoidal 2-functors from them to vector spaces. These three examples are cobordisms, coloured cobordisms and nested cobordisms.

The category of cobordisms is defined in the following way: The objects are closed 1-manifolds and the morphisms are diffeomorphism classes of 2-manifolds whose in-boundary is the first 1-manifold and whose out-boundary the second one. The morphisms of this category are generated under composition and disjoint union by five special cobordisms (cf. [Koc04]). Since some of the cobordisms built this way are equivalent, we also get several relations. A two-dimensional TQFT is a symmetric monoidal functor from the category of cobordisms to the category of vector spaces. This means that we associate a vector space to each circle and a linear map to each cobordism, such that disjoint unions of circles are sent to the tensor products of vector spaces and further conditions are satisfied.

In Khovanov homology, cobordisms, more precisely a two-dimensional TQFT, are used to define a multiplication in a graphically defined algebra, Khovanov's arc algebra H^n [Kho00]. In [Str09] Stroppel generalised this arc algebra by using coloured cobordisms. They have the same generators as the cobordisms, only with the boundaries coloured in three colours. This extended arc algebra is interesting for two reasons: On one hand, it connects Khovanov homology with Lie theory. On the other hand, this algebra allows to extend Khovanov homology to the whole Reshetikhin-Turaev $U_q(\mathfrak{sl}_2)$ tangle invariants.

The nested cobordisms arise in [SW08] where they are used to describe the non-associative multiplication on a convolution algebra. Nested cobordisms keep track of the nestedness of the circles. For that, there is a second pair of pants cobordism, which connects one circle to two nested circles. In [BW09] Beliakova and Wagner consider nested cobordisms and their normal form and show that via the nested cobordisms Khovanov homology arises as well.

A 2-category consists of a category with additional morphisms between the morphisms, called 2-morphisms. We consider our cobordisms as 2-categories by taking the circles as objects, the cobordisms build by the generators as morphisms and the relations as 2-morphisms. The resulting 2-categories are called Cob, $ColCob$ and $NesCob$. As we are working with 2-categories, we consider monoidal 2-functors.

In Section 1 we start by defining semistrict monoidal 2-categories. In Theorem 1.10 we provide the most important technical tool to prove finally in Theorems 1.14, 1.17, 1.32 that

Theorem. *Cob, $ColCob$ and $NesCob$ are semistrict monoidal 2-categories.*

In Section 2 we consider monoidal 2-functors to a 2-category with vector spaces as objects. In Theorems 2.5, 2.6, 2.9 we prove that by specifying certain values on generators we get monoidal 2-functors. In Section 3 we show that the morphisms of our examples have certain normal forms, similar to the normal form of cobordisms that is for example considered in [Koc04].

In the last section of the first part, Section 4, we consider the categories that arise from our three examples by considering the 2-morphisms as relations and dividing them out. We call the resulting categories Cob', $ColCob'$ and $NesCob'$ and get in Theorems 4.7, 4.8, 4.9

Theorem. *Cob', $ColCob'$ and $NesCob'$ are symmetric monoidal categories.*

Introduction to part II

In the second part, we consider irreducible components of special Spaltenstein varieties and generalisations of these. In the end we connect the cohomology of their intersections with a functor from Part I.

For a given nilpotent endomorphism N of $\mathbb{C}^n$, the Springer fibre is the subvariety of the variety of full flags given by the flags fixed under N. If we take partial flags instead of full ones, we get Spaltenstein varieties.

In 1976, Spaltenstein showed that the irreducible components of the Springer fibre are in bijective correspondence with certain tableaux [Spa76]. Furthermore, he stated a bijective correspondence between the irreducible components of Spaltenstein varieties and a subset of the tableaux.

In 2003, Fung considered two special cases of Springer fibres [Fun03]. We are interested in the case where the endomorphism N has at most two Jordan blocks. In this case, he gave an explicit description of irreducible components of the Springer fibre and showed that they are iterated $\mathbb{CP}^1$- bundles. In addition, he used cup diagrams to describe the structure of the irreducible components.

In [SW08], Stroppel and Webster expanded the use of cup diagrams for the description of components of 2-block Springer fibres. Moreover, they introduced what we call generalised irreducible components for Springer fibres. Generalised components are the closure of fixed point attracting cells for a certain torus action. The set of generalised components contains the set of irreducible components. They computed the cohomology of generalised components and their intersections and showed that the intersections are iterated fibre bundles.

In this thesis we consider the special case of 2-block Spaltenstein varieties and generalise the theorems already known for Springer fibres.

We first use results from Spaltenstein's paper to get a bijective correspondence between irreducible components of Spaltenstein varieties and certain standard tableaux in Section 2. Then, in Section 3 we consider the theorem of Fung which explicitly describes the irreducible components of Springer fibres and generalise it to Spaltenstein varieties.

In Section 4 we use fixed points to define generalised irreducible components analogously to what is done for Springer fibres in [SW08]. This results in a description of these generalised components similar to [SW08, Theorem 15]. After that, we give a bijective morphism from the generalised irreducible components of Spaltenstein varieties to those of certain Springer fibres (Theorem 4.8).

Subsequently, in Section 5 we generalise the cup diagrams appearing in [Fun03] and [SW08] by what we call dependence graphs. These dependence graphs consist of labelled and coloured arcs. They describe the structure of generalised irreducible components of Spaltenstein varieties visually (Theorem 5.20) and help to prove some of the following theorems.

Next, we use coloured circle diagrams as in [Str09] to give a condition for the intersection of generalised irreducible components to be empty in section 6. In Section 7 we show, generalising [Fun03] and [SW08], that generalised irreducible components and non-empty intersections of those form iterated fibre bundles, giving a proof that uses cup diagrams. From this we compute the cohomology of the generalised irreducible components and their non-empty intersections using a spectral sequence argument in Section 8.

Finally, in Theorem 8.5 we combine the above to see that we can calculate the cohomology of intersections of the generalised irreducible components $\widetilde{\mathcal{Y}}_w, \widetilde{\mathcal{Y}}_{w'}$ by applying the functor F_{ColCob} from Part I to the circle diagram $CC(w,w')$ associated to the corresponding pairs of row strict tableaux:

Theorem. *The following diagram commutes:*

$$\begin{tikzcd}
\textit{pairs of row strict tableaux} \arrow[r, "(w,w')\mapsto F_{ColCob}(CC(w,w'))"'] \arrow[d] & \textit{vector spaces} \arrow[d, "\mathrm{id}"] \\
\textit{pairs of } \widetilde{\mathcal{Y}}_w \arrow[r, "(\widetilde{\mathcal{Y}}_w,\widetilde{\mathcal{Y}}_{w'})\mapsto H^*(\widetilde{\mathcal{Y}}_w\cap\widetilde{\mathcal{Y}}_{w'})"'] & \textit{vector spaces}
\end{tikzcd}$$

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German introduction

Diese Arbeit besteht aus zwei Teilen verschiedener Natur. Der erste beschäftigt sich mit unterschiedlichen Kobordismus-2-Kategorien und Verallgemeinerungen von zweidimensionalen topologischen Quantenfeldtheorien (TQFTs). Der zweite Teil ist algebro-geometrisch und behandelt spezielle Beispiele von Springerfasern und Spaltensteinvarietäten. Die beiden Teile erhalten am Ende der Arbeit in dem Hauptsatz (Satz 8.5) und Vermutung 8.6 eine Verbindung.

Einleitung zu Teil I

In diesem Teil betrachten und beschreiben wir drei spezielle Beispiele von semistrikt monoidalen 2-Kategorien sowie monoidale 2-Funktoren von diesen zu Vektorräumen. Diese drei Beispiele sind gegeben durch Kobordismen, gefärbte Kobordismen und verschachtelte Kobordismen.

Die Kategorie der Kobordismen ist auf folgende Weise definiert: Die Objekte sind geschlossene 1-Mannigfaltigkeiten und die Morphismen sind Diffeomorphieklassen von 2-Mannigfaltigkeiten, deren eingehender Rand die erste 1-Mannigfaltigkeit ist und deren ausgehender Rand die andere. Die Morphismen dieser Kategorie sind durch Komposition und disjunkte Vereinigung von fünf speziellen Kobordismen erzeugt (vgl. [Koc04]). Da einige der auf diese Weise gebauten Kobordismen äquivalent sind, bekommen wir auch mehrere Relationen. Eine zweidimensionale TQFT ist ein symmetrischer monoidaler Funktor von der Kategorie der Kobordismen in die Kategorie der Vektorräume. Dies bedeutet, dass wir jedem Kreis einen Vektorraum zuordnen und jedem Kobordismus eine lineare Abbildung, so dass disjunkte Vereinigungen von Kreisen auf Tensorprodukte von Vektorräumen geschickt werden und weitere Bedingungen erfüllt sind.

Für Khovanov-Homologie werden Kobordismen, genauer gesagt eine zweidimensionale TQFT, verwendet, um eine Multiplikation in einer grafisch definierten Algebra zu definieren, Khovanovs Arc Algebra H^n [Kho00]. In [Str09] verallgemeinerte Stroppel diese Arc Algebra mit Hilfe von gefärbten Kobordismen. Diese haben die gleichen Erzeuger wie die Kobordismen, nur können die Ränder in drei Farben gefärbt sein. Diese erweiterte Arc Algebra ist aus zwei Gründen interessant: Einerseits verbindet sie Khovanov-Homologie mit Lie-Theorie. Andererseits ermöglicht diese Algebra Khovanov-Homologie zu Reshetikhin-Turaev-$U_q(\mathfrak{sl}_2)$-Tangle-Invarianten zu erweitern.

Die verschachtelten Kobordismen kommen in [SW08] vor, wo sie verwendet werden, um die nicht-assoziative Multiplikation auf einer Konvolutionsalgebra zu beschreiben. Verschachtelte Kobordismen respektieren die Schachtelung der Kreise. Dafür gibt es einen zweiten Hosen-Kobordismus, der einen Kreis mit zwei geschachtelten Kreisen verbindet. In [BW09] betrachten Beliakova und Wagner geschachtelte Kobordismen und ihre Normalform und zeigen, dass auch mit verschachtelten Kobordismen die Khovanov-Homologie entsteht.

Eine 2-Kategorie besteht aus einer Kategorie mit zusätzlichen Morphismen zwischen den Morphismen, genannt 2-Morphismen. Wir betrachten unsere Kobordismen als 2-Kategorien, indem wir die Kreise als Objekte nehmen, die Kobordismen erzeugt durch die

Erzeuger als Morphismen und die Relationen als 2-Morphismen. Die entstehenden 2-Kategorien nennen wir Cob, $ColCob$ und $NesCob$. Da wir mit 2-Kategorien arbeiten, betrachten wir monoidale 2-Funktoren.

In Abschnitt 1 beginnen wir mit der Definition einer semistrikt monoidalen 2-Kategorie. Mit Satz 1.10 stellen wir das wichtigste technische Werkzeug zur Verfügung, um in den Sätzen 1.14, 1.17, 1.32 zu beweisen, dass

Theorem. *Cob, $ColCob$ und $NesCob$ sind semistrikt monoidale 2-Kategorien.*

In Abschnitt 2 betrachten wir monoidale 2-Funktoren in eine 2-Kategorie mit Vektorräumen als Objekte. In den Sätzen 2.5, 2.6, 2.9 beweisen wir, dass wir durch die Vorgabe bestimmter Werte auf Erzeugern monoidale 2-Funktoren erhalten. In Abschnitt 3 zeigen wir, dass die Morphismen in unseren Beispielen bestimmte Normalformen haben, ähnlich wie die Normalform der Kobordismen, die zum Beispiel in [Koc04] beschrieben wird.

Im letzten Abschnitt des ersten Teils, Abschnitt 4, betrachten wir die Kategorien, die aus unseren drei Beispielen entstehen, wenn wir die 2-Morphismen als Relationen auffassen und herausteilen. Wir nennen die entstehenden Kategorien Cob', $ColCob'$ und $NesCob'$ und erhalten in den Sätzen 4.7, 4.8, 4.9:

Theorem. *Cob', $ColCob'$ und $NesCob'$ sind symmetrische monoidale Kategorien.*

Einleitung zu Teil II

Im zweiten Teil betrachten wir irreduzible Komponenten von besonderen Spaltensteinvarietäten und Verallgemeinerungen davon. Am Ende verbinden wir die Kohomologie von ihren Schnitten mit einem Funktor aus Teil I.

Für einen gegebenen nilpotenten Endomorphismus N von $\mathbb{C}^n$ ist die Springerfaser eine Untervarietät der Varietät der vollen Fahnen gegeben durch die Fahnen, die von N fixiert werden. Wenn wir partielle Fahnen anstelle von vollen nehmen, erhalten wir Spaltensteinvarietäten.

Im Jahr 1976 zeigte Spaltenstein, dass die irreduziblen Komponenten der Springerfaser in Bijektion zu bestimmten Tableaus stehen [Spa76]. Darüber hinaus beschrieb er eine Bijektion zwischen den irreduziblen Komponenten von Spaltensteinvarietäten und einer Teilmenge der Tableaus.

Im Jahr 2003 betrachtete Fung zwei besondere Fälle von Springerfasern [Fun03]. Wir interessieren uns für den Fall, in dem der Endomorphismus N höchstens zwei Jordan-Blöcke hat. Fung gab in diesem Fall eine explizite Beschreibung der irreduziblen Komponenten der Springerfaser und zeigte, dass diese iterierte $\mathbb{CP}^1$ -Bündel sind. Außerdem verwendete er Cup-Diagramme, um die Struktur der irreduziblen Komponenten beschreiben.

In [SW08] weiteten Stroppel und Webster den Einsatz von Cup-Diagramme für die Beschreibung der Komponenten von 2-Block Springerfasern aus. Des weiteren definierten sie, was wir als verallgemeinerte irreduzible Komponenten für Springerfasern bezeichnen. Verallgemeinerte Komponenten sind die Abschlüsse von Attraktionszellen von Fixpunkten unter einer bestimmten Toruswirkung. Die Menge der verallgemeinerten Komponenten enthält die Menge der irreduziblen Komponenten. Stroppel und Webster berechneten die Kohomologie von verallgemeinerten Komponenten und deren Schnitten und zeigten, dass die Schnitte iterierte Faserbündel sind.

In dieser Arbeit betrachten wir den speziellen Fall der 2-Block-Spaltensteinvarietäten und verallgemeinern Sätze, die bereits für Springerfasern bekannt sind.

Zuerst benutzen wir Resultate aus Spaltensteins Artikel in Abschnitt 2, um eine Bijektion zwischen irreduziblen Komponenten von Spaltensteinvarietäten und bestimmten Standard-Tableaus herzustellen. Dann, in Abschnitt 3, verallgemeinern wir den Satz von Fung, der die irreduziblen Komponenten von Springerfasern explizit beschreibt, auf Spaltensteinvarietäten.

In Abschnitt 4 verwenden wir Fixpunkte um verallgemeinerte irrreduzible Komponenten zu definieren, analog zur Definition für Springerfasern in [SW08]. Dies führt zu einem Satz, der diese verallgemeinerten Komponenten ähnlich wie [SW08, Satz 15] beschreibt. Danach geben wir einen bijektiven Morphismus vom verallgemeinerten irreduziblen Komponenten von Spaltensteinvarietäten zu denen von bestimmten Springerfasern an (Satz 4.8).

Anschließend, in Abschnitt 5, verallgemeinern wir die Cup-Diagramme aus [Fun03] und [SW08] zu etwas, das wir Abhängigkeitgraphen nennen. Die Abhängigkeitsgraphen bestehen aus beschrifteten farbigen Bögen. Sie beschreiben die Struktur der verallgemeinerten irreduziblen Komponenten von Spaltensteinvarietäten anschaulich (Satz 5.20) und tragen dazu bei, einige der folgenden Sätze zu beweisen.

Als nächstes benutzen wir in Abschnitt 6 farbige Kreisdiagramme wie in [Str09] um eine Bedingung dafür anzugeben, dass die Schnitte von verallgemeinerten irreduziblen Komponenten leer sind. In Abschnitt 7 verallgemeinern wir [Fun03] und [SW08], indem wir zeigen, dass verallgemeinerte irreduzible Komponenten und deren nichtleere Schnitte iterierte Faserbündel sind. Dies beweisen wir mit Hilfe von Cup-Diagrammen. Damit berechnen wir die Kohomologie der verallgemeinerten irreduziblen Komponenten und deren nichtleeren Schnitte mit Hilfe eines Spektralsequenz-Argument in Abschnitt 8.

Schließlich kombinieren wir in Satz 8.5 die obigen Aussagen um zu sehen, dass wir die Kohomologie von Schnitten der verallgemeinerten irreduziblen Komponenten $\widetilde{\mathcal{Y}}_w, \widetilde{\mathcal{Y}}_{w'}$ durch die Anwendung des Funktors F_{ColCob} aus Teil I auf das Kreisdiagramm $CC(w,w')$ der zugehörigen Paare von zeilenstrikten Tableaus berechnen können:

Theorem. *Das folgende Diagramm kommutiert:*

$$\begin{tikzcd}
\textit{Paare von zeilenstrikten Tableaus} \arrow[r, "(w,w')\mapsto F_{ColCob}(CC(w,w'))"'] \arrow[d] & \textit{Vektorräume} \arrow[d, "\mathrm{id}"] \\
\textit{Paare von } \widetilde{\mathcal{Y}}_w \arrow[r, "(\widetilde{\mathcal{Y}}_w,\widetilde{\mathcal{Y}}_{w'})\mapsto H^*(\widetilde{\mathcal{Y}}_w\cap\widetilde{\mathcal{Y}}_{w'})"'] & \textit{Vektorräume}
\end{tikzcd}$$

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Part I

Monoidal 2-functors

1 Semistrict monoidal 2-categories

In this section we first state the definitions we need to finally define a semistrict monoidal 2-category. Then we provide a technical theorem, which helps us to prove that our examples are semistrict monoidal 2-categories.

Notation 1.1. In this part, we write compositions in the following order: $f : A \to B, g : B \to C$, then $fg : A \to C$.

Definition 1.2. ([Wei94]) A *category* $\mathcal{C}$ consists of the following:

- a class $ob(\mathcal{C})$ of objects,
- a set $Hom_{\mathcal{C}}(A,B)$ of morphisms for every ordered pair (A,B) of objects,
- an identity morphism $\mathrm{id}_A \in Hom_{\mathcal{C}}(A,A)$ for each object A,
- a composition function $Hom_{\mathcal{C}}(A,B) \times Hom_{\mathcal{C}}(B,C) \to Hom_{\mathcal{C}}(A,C)$ for every ordered triple (A,B,C) of objects.

We write $f : A \to B$ to indicate that f is a morphism in $Hom_{\mathcal{C}}(A,B)$, and we write fg or $f \circ g$ for the composition of $f : A \to B$ with $g : B \to C$. The above data is subject to two axioms:

- Associativity Axiom: $f(gh) = (fg)h$ for $f : A \to B$, $g : B \to C$, $h : C \to D$
- Unit Axiom: $f \circ \mathrm{id}_B = f = \mathrm{id}_A \circ f$ for $f : A \to B$

A category $\mathcal{C}$ is *small* if $ob(\mathcal{C})$ is a set.

The following motivation and definition of a 2-category can be found for example in [ML98]. In a 2-category we have not only objects and morphisms but morphisms between some morphisms, called 2-morphisms. If we have a morphism $\alpha : f \Rightarrow g$, both morphisms are required to have the same source and target object. Such a 2-morphism is often displayed as:

$$\begin{tikzcd} \blacksquare \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & \blacksquare \end{tikzcd}$$

Thus, there are 2 different possibilities of compositions of 2-morphisms, the *vertical composition*:

$$\begin{tikzcd} \blacksquare \arrow[r, bend left=60, "f"] \arrow[r] \arrow[r, bend right=60, "g"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \blacksquare \end{tikzcd}$$

and the *horizontal composition*:

$$\begin{tikzcd} \blacksquare \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & \blacksquare \arrow[r, bend left=50, "f'"] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & \blacksquare \end{tikzcd}$$

Notation 1.3. We denote the composition of morphisms and the horizontal composition of 2-morphisms by $\circ$ or simply by juxtaposition, and we denote the vertical composition of 2-morphisms by $\bullet$. Here we denote the composition of morphisms in the same way as the horizontal composition of 2-morphisms because this makes it easier to consider a morphism as the associated identity 2-morphism.

For a 2-category, we require the objects and morphisms to build a category. In addition, the objects and the 2-morphisms are to build a category under horizontal composition. In particular, there are identity 2-morphisms for this composition, $1_{\mathrm{id}_A} : \mathrm{id}_A \Rightarrow \mathrm{id}_A : A \to A$. Furthermore, if we fix a source and target, the morphisms between them and the 2-morphisms between this morphisms are to build a category under vertical composition. In particular, there are identity 2-morphisms $1_f : f \Rightarrow f$. Moreover, we have conditions which link the horizontal composition and the vertical composition: For the identities we require $1_{f\circ f'} = 1_f \circ 1_{f'}$. For

$$\alpha : f \Rightarrow g : A \to B, \quad \alpha' : f' \Rightarrow g' : B \to C,$$
$$\beta : g \Rightarrow h : A \to B, \quad \beta' : g' \Rightarrow h' : B \to C$$

we require

$$(\alpha \circ \alpha') \bullet (\beta \circ \beta') = (\alpha \bullet \beta) \circ (\alpha' \bullet \beta') : f \circ f' \Rightarrow h \circ h' : A \to C.$$

This last condition is called "middle four exchange" and pictured as

$$\begin{tikzcd}
\blacksquare \arrow[r, bend left=60, "f"] \arrow[r] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \blacksquare \arrow[r, bend left=60, "f'"] \arrow[r] \arrow[r, bend right=60, "h'"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha'"] \arrow[r, phantom, shift right=3, "\Downarrow\beta'"] & \blacksquare
\end{tikzcd}$$

By [ML98] we can summarise this in the following equivalent and more compact definition:

Definition 1.4. ([ML98])
A *2-category* is given by the following data:

i) A class of objects $A, B, C, \dots$

ii) A function which assigns to each ordered pair of objects (A, B) a category $T(A, B)$, called *vertical category*.

iii) For each ordered triple (A, B, C) of objects a functor

$$K_{A,B,C} : T(A, B) \times T(B, C) \to T(A, C)$$

called *composition*.

iv) For each object A a functor

$$U_A : \mathbf{1} \to T(A, A)$$

where $\mathbf{1}$ is the category with one object and one morphism.

This data is required to satisfy the associative law for the composition iii) and the requirement that U_A provides both a left and right identity for this composition.

Example 1.5. An example of a 2-category is given by the following: The objects are small categories, the morphism are functors and the 2-morphism are natural transformations. (see e.g. [ML98, p. 272])

Our next goal is to define a tensor product in our category. The next definition is fundamental for this and will be of importance in the next section as well:

Definition 1.6. ([Kel74]) Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories. A *2-functor* $F : \mathcal{C} \to \mathcal{D}$ sends objects of $\mathcal{C}$ to objects of $\mathcal{D}$, morphisms of $\mathcal{C}$ to morphisms of $\mathcal{D}$ and 2-morphisms of $\mathcal{C}$ to 2-morphisms of $\mathcal{D}$, respectively, preserving domains and codomains and identities and compositions of all kinds.

Now can describe how to define a tensor product on our 2-category:

Definition 1.7. ([Lau05])
A *semistrict monoidal 2-category* consists of a 2-category $\mathcal{C}$ together with:

1) An object $I \in \mathcal{C}$ called the *unit*.

2) For any two objects $A, B \in \mathcal{C}$ an object $A \otimes B$ in $\mathcal{C}$, called the *tensor product*.

3) For any 1-morphism $f: A \to A'$ and any object $B \in \mathcal{C}$ 1-morphisms $f \otimes B: A \otimes B \to A' \otimes B$ and $B \otimes f: B \otimes A \to B \otimes A'$.

4) For any object $B \in \mathcal{C}$ and any 2-morphism $\alpha: f \Rightarrow f'$ 2-morphisms $\alpha \otimes B: f \otimes B \Rightarrow f' \otimes B$ and $B \otimes \alpha: B \otimes f \Rightarrow B \otimes f'$.

5) For any two 1-morphisms $f: A \to A'$ and $g: B \to B'$ a 2-isomorphism (i.e. a 2-morphism with an inverse) $\bigotimes_{f,g} : (A \otimes g) \circ (f \otimes B') \Rightarrow (f \otimes B) \circ (A' \otimes g)$ which we display by the following diagram

$$\begin{tikzcd}
A \otimes B \arrow[r, "A\otimes g"] \arrow[d, "f\otimes B"'] \arrow[dr, phantom, "\Downarrow \bigotimes_{f,g}"] & A \otimes B' \arrow[d, "f\otimes B'"] \\
A' \otimes B \arrow[r, "A'\otimes g"'] & A' \otimes B'
\end{tikzcd}$$

This data is subject to the following conditions.

i) For any object $A \in \mathcal{C}$ we have that $A \otimes -: \mathcal{C} \to \mathcal{C}$ and $- \otimes A: \mathcal{C} \to \mathcal{C}$ are 2-functors.

ii) For x any object, morphism or 2-morphism of $\mathcal{C}$ we have $x \otimes I = I \otimes x = x$.

iii) For x any object, morphism or 2-morphism of $\mathcal{C}$, and for all objects $A, B \in \mathcal{C}$ we have $A \otimes (B \otimes x) = (A \otimes B) \otimes x$, $A \otimes (x \otimes B) = (A \otimes x) \otimes B$ and $x \otimes (A \otimes B) = (x \otimes A) \otimes B$.

iv) For arbitrary 1-morphisms $f: A \to B$, $g: B \to B'$ and $h: C \to C'$ in $\mathcal{C}$ we have $\bigotimes_{A\otimes g,h} = A \otimes \bigotimes_{g,h}$, $\bigotimes_{f\otimes B,h} = \bigotimes_{f,B\otimes h}$ and $\bigotimes_{f,g\otimes C} = \bigotimes_{f,g} \otimes C$.

v) For any objects $A, B \in \mathcal{C}$ we have $\mathrm{id}_A \otimes B = A \otimes \mathrm{id}_B = \mathrm{id}_{A\otimes B}$, and for any 1-morphism $f: A \to A'$, $g: B \to B'$ in $\mathcal{C}$ we have $\bigotimes_{\mathrm{id}_A, g} = 1_{A\otimes g}$ and $\bigotimes_{f,\mathrm{id}_B} = 1_{f\otimes B}$.

vi) For any 1-morphisms $f, h: A \to A'$, $g, k: B \to B'$, and any 2-morphisms $\alpha: f \Rightarrow h$, and $\beta: g \Rightarrow k$, we have the equality of 2-morphisms

$$\begin{aligned}
{\bigotimes_{f,g}}^{-1} \bullet ((A \otimes \beta) \circ (\alpha \otimes B')) = ((\alpha \otimes B) \circ (A' \otimes \beta)) \bullet {\bigotimes_{h,k}}^{-1} \\
: (f \otimes B) \circ (A' \otimes g) \Rightarrow (A \otimes k) \circ (h \otimes B')
\end{aligned}$$

displayed by the following diagrams

$$\begin{tikzcd}
A \otimes B \arrow[r, "f\otimes B"] \arrow[d, bend right=50, "A\otimes k"'] \arrow[d, bend left=50, "A\otimes g"] \arrow[d, phantom, "\Leftarrow A\otimes\beta"] \arrow[dr, phantom, "\Downarrow \bigotimes^{-1}_{f,g}"] & A' \otimes B \arrow[d, "A'\otimes g"] \\
A \otimes B' \arrow[r, bend left, "f\otimes B'"] \arrow[r, bend right, "h\otimes B'"'] \arrow[r, phantom, "\Downarrow \alpha\otimes B'"] & A' \otimes B'
\end{tikzcd}
=
\begin{tikzcd}
A \otimes B \arrow[r, bend left, "f\otimes B"] \arrow[r, bend right, "h\otimes B"'] \arrow[r, phantom, "\Downarrow \alpha\otimes B"] \arrow[d, "A\otimes k"'] \arrow[dr, phantom, "\bigotimes^{-1}_{h,k}\Downarrow"] & A' \otimes B \arrow[d, bend right=50, "A'\otimes k"'] \arrow[d, bend left=50, "A'\otimes g"] \arrow[d, phantom, "\Leftarrow A'\otimes\beta"] \\
A \otimes B' \arrow[r, "h\otimes B'"'] & A' \otimes B'
\end{tikzcd}$$

vii) For any 1-morphisms $f\colon A\to A'$, $g\colon B\to B'$, $f'\colon A'\to A''$, $g'\colon B'\to B''$ we have an equality of 2-morphisms

$$(1_{A\otimes g}\circ\otimes_{f,g'}\circ 1_{f'\otimes B''})\bullet(\otimes_{f,g}\circ\otimes_{f',g'})\bullet(1_{f\otimes B}\circ\otimes_{f',g}\circ 1_{A''\otimes g'})=\otimes_{ff',gg'}$$

displayed by the diagrams

$$\begin{tikzcd}
A\otimes B \arrow[r, "A\otimes g"] \arrow[d, "f\otimes B"'] \arrow[dr, phantom, "\Downarrow\otimes_{f,g}"] & A\otimes B' \arrow[r, "A\otimes g'"] \arrow[d, "f\otimes B'"] \arrow[dr, phantom, "\Downarrow\otimes_{f,g'}"] & A\otimes B'' \arrow[d, "f\otimes B''"] \\
A'\otimes B \arrow[r, "A'\otimes g"] \arrow[d, "f'\otimes B"'] \arrow[dr, phantom, "\Downarrow\otimes_{f',g}"] & A'\otimes B' \arrow[r, "A'\otimes g'"] \arrow[d, "f'\otimes B'"] \arrow[dr, phantom, "\Downarrow\otimes_{f',g'}"] & A'\otimes B'' \arrow[d, "f'\otimes B''"] \\
A''\otimes B \arrow[r, "A''\otimes g"] & A''\otimes B' \arrow[r, "A''\otimes g'"] & A''\otimes B''
\end{tikzcd}
=
\begin{tikzcd}
A\otimes B \arrow[r, "A\otimes g\circ g'"] \arrow[d, "f\circ f'\otimes B"'] \arrow[dr, phantom, "\Downarrow\otimes_{f\circ f',g\circ g'}"] & A\otimes B'' \arrow[d, "f\circ f'\otimes B''"] \\
A''\otimes B \arrow[r, "A''\otimes g\circ g'"'] & A''\otimes B''
\end{tikzcd}.$$

Now we want to construct a semistrict monoidal 2-category by giving generators for the objects, 1-morphisms and 2-morphisms, which satisfy certain conditions.

Definition 1.8. Assume $\mathcal{C}$ consists of a set of objects $\mathcal{O}$, a set of morphisms $\mathcal{M}$ and a set of 2-morphisms $\mathcal{N}$ together with

- a product $\mathcal{O}\otimes\mathcal{O}\to\mathcal{O}$, $(A,B)\mapsto A\otimes B$
- a product $\mathcal{M}\otimes\mathcal{M}\to\mathcal{M}$, $(f,g)\mapsto f\otimes g$
- a product $\mathcal{N}\otimes\mathcal{N}\to\mathcal{N}$, $(\alpha,\beta)\mapsto\alpha\otimes\beta$
- a horizontal composition $\mathcal{M}\otimes\mathcal{M}\to\mathcal{M}$, $(f,g)\mapsto f\circ g$
- a horizontal composition $\mathcal{N}\otimes\mathcal{N}\to\mathcal{N}$, $(\alpha,\beta)\mapsto\alpha\circ\beta$
- a vertical composition $\mathcal{N}\otimes\mathcal{N}\to\mathcal{N}$, $(\alpha,\beta)\mapsto\alpha\bullet\beta$

Assume further there is an object I in $\mathcal{C}$, for every object A there is a morphism called id_A and for any morphism f there is a 2-morphism called 1_f.

Then we say $\mathcal{C}$ *is generated by* the basic objects $A_1,A_2\ldots$, basic morphisms $f_1,f_2,\ldots$ and basic 2-morphisms $\alpha_1,\alpha_2\ldots$ if

(I) each object in $\mathcal{C}$ is the finite product of basic objects, where I is the empty product.

(II) each morphism in $\mathcal{C}$ is obtained from the basic morphisms and the id_{A_i} by finitely many products and compositions.

(III) each 2-morphism in $\mathcal{C}$ is obtained from the basic 2-morphisms and the 1_f for f a morphism by finitely many products, horizontal compositions and vertical compositions.

Remark 1.9. Strictly speaking, one has to define the notion of "generating" in the previous definition as follows:
$(A, B, \Phi_1, \dots, \Phi_l)$ is a set with distinguished basic elements and products if the following holds: We have $B \subset A$, where B is the set of basic elements, and

$$\Phi_j : \underbrace{B \times \dots \times B}_{m_j} \to A.$$

A is generated by B via $\Phi_1, \dots, \Phi_l$, if for all $a \in A$ we have

$$a = \Psi_{i_1} \circ \dots \circ \Psi_{i_k}(b_{l_1}, \dots, b_{l_n})$$

where $b_{l_1}, \dots, b_{l_n} \in B$ and $\Psi_j := 1 \times \dots \times 1 \times \Phi_j \times 1 \times \dots \times 1$.

For showing that our examples in the next sections are semistrict monoidal 2-categories we will mainly use the following theorem:

Theorem 1.10. *Let $\mathcal{C}$ be generated by the basic objects $A_1, A_2 \dots$, basic morphisms $f_1, f_2, \dots$ and basic 2-morphisms $\alpha_1, \alpha_2 \dots$ in $\mathcal{C}$ and assume the following holds (where the product is denoted by $\otimes$):*

a) For $A = A_{i_1} \otimes \dots \otimes A_{i_k}$ an object in $\mathcal{C}$ we have $\mathrm{id}_A = \mathrm{id}_{A_{i_1}} \otimes \dots \otimes \mathrm{id}_{A_{i_k}}$.

b) Compositions of morphisms and 2-morphisms are associative.

c) We have $f_i \,\mathrm{id}_B = f_i = \mathrm{id}_A f_i$ for all $f_i : A \to B$ and $\mathrm{id}_{A_i} \circ \mathrm{id}_{A_i} = \mathrm{id}_{A_i}$, $i = 1, 2 \dots$.

d) For any 2-morphism $\alpha : f \Rightarrow g : A \to B$ and any morphisms $f, g : A \to B$, $h : B \to C$ we have $1_{\mathrm{id}_A} \alpha = \alpha = \alpha 1_{\mathrm{id}_B}$, $1_f \bullet \alpha = \alpha = \alpha \bullet 1_g$, $1_{gh} = 1_g 1_h$.

e) For any 2-morphisms $\alpha : f \Rightarrow g : A \to B$, $\alpha' : f' \Rightarrow g' : B \to C$, $\beta : g \Rightarrow h : A \to B$ and $\beta' : g' \Rightarrow h' : B \to C$ we have

$$(\alpha \circ \alpha') \bullet (\beta \circ \beta') = (\alpha \bullet \beta) \circ (\alpha' \bullet \beta').$$

f) We have $A \otimes I = I \otimes A = A$ for A any object, $f \otimes \mathrm{id}_I = \mathrm{id}_I \otimes f = f$ for any morphism f and $\alpha \otimes 1_{\mathrm{id}_A} = 1_{\mathrm{id}_A} \otimes \alpha = \alpha$ for any 2-morphism α of $\mathcal{C}$.

g) For any morphisms $f : A \to B$ and $g : C \to D$ we have $1_f \otimes 1_g = 1_{f \otimes g}$.

h) For any morphisms $f : A \to B$, $h : B \to C$, $g : D \to E$, $k : E \to F$ we have $(f \otimes g) \circ (h \otimes k) = (f \circ h) \otimes (g \circ k)$.

i) For any 2-morphisms $\alpha : f \Rightarrow g : A \to B$, $\alpha' : f' \Rightarrow g' : B \to C$, $\beta : h \Rightarrow k : D \to E$ and $\beta' : h' \Rightarrow k' : E \to F$ we have $(\alpha \otimes \beta) \circ (\alpha' \otimes \beta') = (\alpha \circ \alpha') \otimes (\beta \circ \beta')$.

j) For any 2-morphisms $\alpha : f \Rightarrow g : A \to B$, $\alpha' : g \Rightarrow h : A \to B$, $\beta : f' \Rightarrow g' : C \to D$ and $\beta' : g' \Rightarrow h' : C \to D$ we have $(\alpha \otimes \beta) \bullet (\alpha' \otimes \beta') = (\alpha \bullet \alpha') \otimes (\beta \bullet \beta')$.

k) The product of objects, morphisms or 2-morphisms is associative.

Then $\mathcal{C}$ is a semistrict monoidal 2-category with the product as tensor product.

Proof. As a tensor product we take the product existing by assumption.

By a) we have

$$\mathrm{id}_A \otimes \mathrm{id}_B = \mathrm{id}_{A\otimes B} \tag{1.1}$$

for all objects in $\mathcal{C}$.

$\mathcal{C}$ is a 2-category:

By b) the composition of morphisms is associative. If we have $f \circ \mathrm{id}_A = f$ and $g \circ \mathrm{id}_B = g$, by h) we get

$$(f \otimes g) \circ \mathrm{id}_{A\otimes B} \overset{(1.1)}{=} (f \otimes g) \circ (\mathrm{id}_A \otimes \mathrm{id}_B) = (f \circ \mathrm{id}_A) \otimes (g \circ \mathrm{id}_B) = f \otimes g$$

for $f : A' \to A$ and $g : B' \to B$, and the same for the composition on the other side follows analogously. In particular we get $\mathrm{id}_A \circ \mathrm{id}_A = \mathrm{id}_A$. Moreover, we have $f \circ f' \circ \mathrm{id}_{A''} = f \circ f' = \mathrm{id}_A \circ f \circ f'$ if $f : A \to A'$, $f' : A' \to A''$ and $\mathrm{id}_A f = f$, $f' \mathrm{id}_{A''} = f'$ holds. Thus by c) we get $f \mathrm{id}_B = f = \mathrm{id}_A f$ for all morphisms $f : A \to B$. Therefore, the objects and the morphisms of $\mathcal{C}$ form a category.

By b) the horizontal composition of 2-morphisms is associative, and by d) we have $1_{\mathrm{id}_A}\alpha = \alpha = \alpha 1_{\mathrm{id}_B}$. So the 2-morphisms with horizontal composition form a category.

By b) the vertical composition of 2-morphisms is associative, and by d) we have $1_f \bullet \alpha = \alpha = \alpha \bullet 1_g$. Thus, the morphisms between fixed source and target and the 2-morphisms with vertical composition form a 2-category. Furthermore, by d) we have $1_{fg} = 1_f 1_g$, and by e) the "middle four exchange" holds. So, the conditions of a 2-category are fulfilled.

$\mathcal{C}$ is a semistrict monoidal 2-category:

We check the conditions of Definition 1.7:

1) This is given by assumption.

2) We have $A \otimes B \in \mathcal{C}$, since by definition A and B are tensor products of basic objects, thus $A \otimes B$ is also a tensor product of basic objects.

3) $f \otimes B := f \otimes \mathrm{id}_B$ and $B \otimes f := \mathrm{id}_B \otimes f$ are in $\mathcal{C}$ by definition.

4) $(\alpha \otimes B : f \otimes B \Rightarrow f' \otimes B) := (\alpha \otimes 1_{\mathrm{id}_B} : f \otimes \mathrm{id}_B \Rightarrow f' \otimes \mathrm{id}_B)$ and $(B \otimes \alpha : B \otimes f \Rightarrow B \otimes f') := (1_{\mathrm{id}_B} \otimes \alpha : \mathrm{id}_B \otimes f \Rightarrow \mathrm{id}_B \otimes f')$ are 2-morphisms by definition.

5) We have

$$\begin{aligned} & \bigotimes_{f,g} : (A \otimes g) \circ (f \otimes B') \Rightarrow (f \otimes B) \circ (A' \otimes g) \\ = \quad & \bigotimes_{f,g} : (\mathrm{id}_A \otimes g) \circ (f \otimes \mathrm{id}_{B'}) \Rightarrow (f \otimes \mathrm{id}_B) \circ (\mathrm{id}_{A'} \otimes g) \\ \overset{h)}{=} \quad & \bigotimes_{f,g} : (\mathrm{id}_A \circ f) \otimes (g \circ \mathrm{id}_{B'}) \Rightarrow (f \circ \mathrm{id}_{A'}) \otimes (\mathrm{id}_B \circ g) \\ = \quad & \bigotimes_{f,g} : f \otimes g \Rightarrow f \otimes g. \end{aligned}$$

$\bigotimes_{f,g} := 1_{f \otimes g}$ is a 2-morphism and even an 2-isomorphism, since it is an identity.

i) $A \otimes \_$ is a 2-functor:
$A \otimes \_$ preserves domains and codomains of morphisms and 2-morphisms by 3) and 4). Furthermore, we have

$$A \otimes f \circ g = \mathrm{id}_A \otimes (f \circ g) \overset{h)}{=} (\mathrm{id}_A \otimes f) \circ (\mathrm{id}_A \otimes g) = (A \otimes f) \circ (A \otimes g)$$

and

$$A \otimes \mathrm{id}_C = \mathrm{id}_A \otimes \mathrm{id}_C \overset{(1.1)}{=} \mathrm{id}_{A \otimes C},$$

hence functoriality for morphisms holds.

The functoriality holds for 2-morphisms as well: Let $\alpha : f \Rightarrow g : B \to C$, $\alpha' : f' \Rightarrow g' : C \to D$. We have

$$\begin{aligned} (A \otimes \alpha)(A \otimes \alpha') = (1_{\mathrm{id}_A} \otimes \alpha)(1_{\mathrm{id}_A} \otimes \alpha') & \overset{i)}{=} (1_{\mathrm{id}_A} 1_{\mathrm{id}_A}) \otimes (\alpha\alpha') \\ & \overset{d)}{=} 1_{\mathrm{id}_A \circ \mathrm{id}_A} \otimes (\alpha\alpha') = 1_{\mathrm{id}_A} \otimes (\alpha\alpha') = A \otimes (\alpha\alpha'). \end{aligned}$$

Let $\alpha : f \Rightarrow g : B \to C, \beta : g \Rightarrow h : B \to C$. We have

$$A \otimes (\alpha \bullet \beta) = (1_{\mathrm{id}_A} \bullet 1_{\mathrm{id}_A}) \otimes (\alpha \bullet \beta) \overset{j)}{=} (A \otimes \alpha) \bullet (A \otimes \beta) \text{ and}$$

$$A \otimes 1_f = 1_{\mathrm{id}_A} \otimes 1_f \overset{g)}{=} 1_{\mathrm{id}_A \otimes f} = 1_{A \otimes f}.$$

Analogously, $\_ \otimes A$ is a 2-functor.

ii) This holds by f).

iii) By k) the tensor product of objects, morphisms and 2-morphisms is associative, in particular if some of them are identity morphisms or identity 2-morphisms.

iv) We have $\bigotimes_{A \otimes g,h} = 1_{(A \otimes g) \otimes h} \overset{k)}{=} 1_{\mathrm{id}_A \otimes g \otimes h} \overset{g)}{=} 1_{\mathrm{id}_A} \otimes 1_{g \otimes h} = A \otimes \bigotimes_{g,h}$.
Analogously, we have $\bigotimes_{f,g \otimes C} = \bigotimes_{f,g} \otimes C$.
Furthermore, $\bigotimes_{f \otimes B,h} = 1_{(f \otimes \mathrm{id}_B) \otimes h} \overset{k)}{=} 1_{f \otimes (\mathrm{id}_B \otimes h)} = \bigotimes_{f,B \otimes h}$.

v) We have $\mathrm{id}_A \otimes B = \mathrm{id}_A \otimes \mathrm{id}_B \overset{(1.1)}{=} \mathrm{id}_{A\otimes B} = \mathrm{id}_A \otimes \mathrm{id}_B = A \otimes \mathrm{id}_B$ and $\bigotimes_{\mathrm{id}_A,g} = 1_{\mathrm{id}_A \otimes g} = 1_{A\otimes g}$. Analogously, $\bigotimes_{f,\mathrm{id}_B} = 1_{f\otimes B}$.

vi) We have $\bigotimes^{-1}_{f,g} \bullet ((A\otimes \beta)\circ(\alpha\otimes B')) \overset{i)}{=} \bigotimes^{-1}_{f,g} \bullet (1_{\mathrm{id}_A}\alpha \otimes \beta 1_{\mathrm{id}_{B'}}) = 1_{f\otimes g} \bullet (\alpha \otimes \beta)$ $\overset{d)}{=} (\alpha\otimes\beta)\bullet 1_{h\otimes k} = (\alpha 1_{\mathrm{id}_{A'}} \otimes 1_{\mathrm{id}_B}\beta)\bullet \bigotimes^{-1}_{h,k} \overset{i)}{=} ((\alpha\otimes B)\circ(A'\otimes\beta))\bullet\bigotimes^{-1}_{h,k}$.

vii) We have

$$\begin{aligned}
&(1_{A\otimes g}\circ \textstyle\bigotimes_{f,g'} \circ 1_{f'\otimes B''})\bullet(\textstyle\bigotimes_{f,g}\circ\textstyle\bigotimes_{f',g'})\bullet(1_{f\otimes B}\circ\textstyle\bigotimes_{f',g}\circ 1_{A''\otimes g'})\\
&\quad = (1_{A\otimes g}\circ 1_{f\otimes g'}\circ 1_{f'\otimes B''})\bullet(1_{f\otimes g}\circ 1_{f'\otimes g'})\bullet(1_{f\otimes B}\circ 1_{f'\otimes g}\circ 1_{A''\otimes g'})\\
&\quad \overset{d),h)}{=} 1_{\mathrm{id}_A ff'\otimes gg'\mathrm{id}_{B''}}\bullet 1_{ff'\otimes gg'}\bullet 1_{ff'\mathrm{id}_{A''}\otimes \mathrm{id}_B gg'}\\
&\quad = 1_{ff'\otimes gg'}\bullet 1_{ff'\otimes gg'}\bullet 1_{ff'\otimes gg'}\\
&\quad \overset{d)}{=} 1_{ff'\otimes gg'} = \textstyle\bigotimes_{ff',gg'}.
\end{aligned}$$

□

Corollary 1.11. *Let $\mathcal{C}$ be a 2-category together with a tensor product on objects, morphisms and 2-morphisms satisfying the following conditions:*

f) There is an object $I\in\mathcal{C}$ such that we have $A\otimes I = I\otimes A = A$ for A any object, $f\otimes \mathrm{id}_I = \mathrm{id}_I \otimes f = f$ for any morphism f and $\alpha\otimes 1_{\mathrm{id}_A} = 1_{\mathrm{id}_A}\otimes\alpha = \alpha$ for any 2-morphism α of $\mathcal{C}$

g) For any morphisms $f: A\to B$ and $g: C\to D$ we have $1_f\otimes 1_g = 1_{f\otimes g}$

h) For any morphisms $f: A\to B$, $h: B\to C$, $g: D\to E$, $k: E\to F$ we have $(f\otimes g)\circ(h\otimes k) = (f\circ h)\otimes(g\circ k)$

i) For any 2-morphisms $\alpha: f\Rightarrow g: A\to B$, $\alpha': f'\Rightarrow g': B\to C$, $\beta: h\Rightarrow k: D\to E$ and $\beta': h'\Rightarrow k': E\to F$ we have $(\alpha\otimes\beta)\circ(\alpha'\otimes\beta') = (\alpha\circ\alpha')\otimes(\beta\circ\beta')$

j) For any 2-morphisms $\alpha: f\Rightarrow g: A\to B$, $\alpha': g\Rightarrow h: A\to B$, $\beta: f'\Rightarrow g': C\to D$ and $\beta': g'\Rightarrow h': C\to D$ we have $(\alpha\otimes\beta)\bullet(\alpha'\otimes\beta') = (\alpha\bullet\alpha')\otimes(\beta\bullet\beta')$

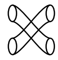
k) The tensor product of objects, morphisms or 2-morphisms is associative

l) For any objects A, B we have $\mathrm{id}_A\otimes\mathrm{id}_B = \mathrm{id}_{A\otimes B}$

Then $\mathcal{C}$ is a semistrict monoidal 2-category.

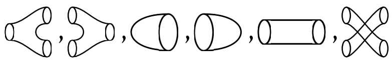
Proof. In the part of the proof of Theorem 1.10, where semistrict monoidal is shown, we only use the equalities coming from being a 2-category and the conditions f) to k) and the equality (1.1). However, the conditions f) to k) are also given here, and the equality (1.1) is given as the additional condition l). □

1.1 Cob

Remark 1.12. In the following, by the picture  we mean that we do not distinguish between top and bottom tube.

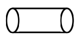
Cob:
Cob is generated by

- the circle O as basic object,
- the following basic morphisms

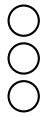
  

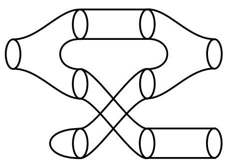
  which we call basic cobordisms,
- the basic relations pictured in Section A.1 as basic 2-morphisms,

where the tensor products, horizontal and vertical compositions are given as follows:
The product for objects and morphisms is the disjoint union i.e. the side by side writing. By the tensor product of 2-morphisms we mean that we apply one 2-morphism to one part of a tensor product of morphisms and the other one to the other part of the tensor product. The horizontal composition of morphisms is given by putting the cobordisms after each other. The horizontal composition of 2-morphisms is the application of one 2-morphism to one part of a composition of morphisms and the other 2-morphisms to the other part. By the vertical composition of 2-morphisms we mean applying the first 2-morphism to a morphism and the second one to the target of the first 2-morphism.

For $A = O \otimes \cdots \otimes O$ an object in $\mathcal{C}$ we define $\mathrm{id}_A := \mathrm{id}_O \otimes \cdots \otimes \mathrm{id}_O$, where  $= \mathrm{id}_O$, and we define 1_f = "apply no relation".

In addition the morphisms are considered modulo the local relations given by the identity relations pictured in A.1.

Example 1.13. An example for an object in *Cob*: 

An example for a morphism in *Cob*: 

The following theorem is a folklore result, see e.g. [Gau03, 5].

Theorem 1.14. *Cob is a semistrict monoidal 2-category.*

Proof. We check the conditions of Theorem 1.10:
By definition Cob is generated in the way required by the theorem. We have a) by definition. c) follows from the identity relations. Since the composition is defined as writing one after another, b) follows for the composition of morphisms and the horizontal composition of 2-morphisms. By definition b) holds also for the vertical composition of 2-morphisms. If a morphism consists of two parts one put (horizontally) after the other and we apply 2-morphisms to one part, then this is independent of what we apply to the other part. Thus, e) holds. Since we defined 1_f = "apply no relation", we immediately have d) and g).

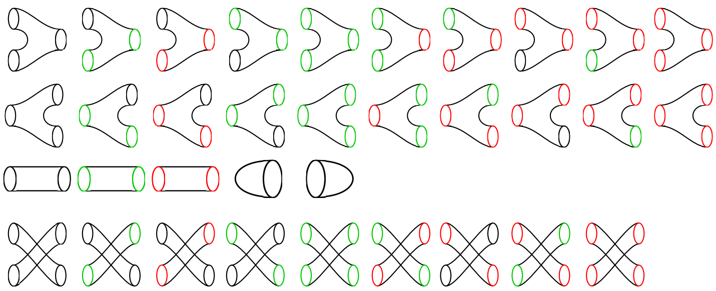
As a disjoint union, the tensor product of objects, morphism and 2-morphisms is associative, thus k) holds. By the definition of the composition and the tensor product of morphisms as putting after each other and putting next to each other, h) follows; analogously i) and j) follow from the definitions of these concatenations. The domains and codomains fit by definition of the concatenations.

We define I to be the empty set. Then we have $A \otimes I = A = I \otimes A$, $\mathrm{id}_I \otimes f = f = f \otimes \mathrm{id}_I$ and $1_{\mathrm{id}_I} \otimes \alpha = \alpha = \alpha \otimes 1_{\mathrm{id}_I}$, hence f) holds. □

1.2 ColCob

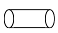
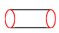
ColCob:
$ColCob$ is generated by

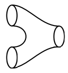
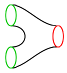
- the black circle, the green circle and the red circle as basic objects,
- the following basic morphisms



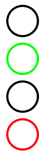
which we call again basic cobordisms,

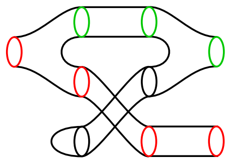
- the basic relations pictured in Section A.2 as basic 2-morphisms,

where the tensor products, horizontal and vertical compositions are given as for Cob. For A_{i_j} basic objects and $A = A_{i_1} \otimes \cdots \otimes A_{i_k}$ an object in $ColCob$ we define $\mathrm{id}_A := \mathrm{id}_{A_{i_1}} \otimes \cdots \otimes \mathrm{id}_{A_{i_k}}$, where $id_B :=$ , $id_G :=$  and $id_R :=$  for B, G, R a black, green, red circle, respectively. We define 1_f := "apply no relation" for a morphism f. Again, the morphisms are considered modulo the local relations given by the identity relations pictured in A.2.

Remark 1.15. Here the colourings of the pair of pants cobordisms are given by the following rule: The color red is stronger than the color green and both are stronger than the color black. So if one of the circles of the in-boundary of a  is red, then the out--boundary is red. If one of the circles of the in-boundary is green and the other is black or green, then the out-boundary is green. In addition there is the cobordism . These colourings are motivated by [Str09, 5.4] where the colourings of the circles correspond to the number of outer points of circles in circle diagrams as in (II.6). The coloured pair of pants cobordisms are used to go from two of these circles to one or vice versa.

The relations for $ColCob$ consist of all the fitting colourings of the relations for Cob, i.e. if the cobordisms of a relation can both be coloured such that the basic cobordisms they consist of are in the list and the boundaries are coloured in the same way, then the relation exists in this colouring.

Example 1.16. An example for an object in $ColCob$: 

An example for a morphism in $ColCob$: 

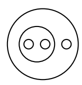
Theorem 1.17. *$ColCob$ is a semistrict monoidal 2-category.*

Proof. As Cob, by definition $ColCob$ is generated in the way required by Theorem 1.10 and condition a) holds. Again, we define I as the empty set. For morphisms b), c), f), h) and k) hold for the same reasons as in the proof for Cob, since the colours only constrain the possible compositions. On the other hand, if it is possible to compose the morphisms, then the same equations as for Cob hold.

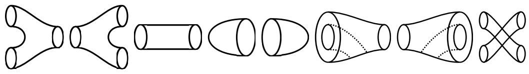
For 2-morphisms we have b), d), e), f), g), i), j) and k) analogously as in the proof for Cob, because for the 2-morphisms in $ColCob$ the same equations hold as for the associated colourless 2-morphisms in Cob. □

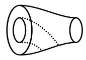
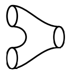
1.3 NesCob

Definition 1.18. A *circle constellation* is a collection of non-intersecting circles with one distinguished circle containing all other circles. The distinguished circle is denoted $o(A)$ for a circle constellation A and called the *outer circle*. For a collection of circle constellations we denote by $o(A)$ the set of all its outer circles, i.e. all the outer circles of the individual circle constellations.

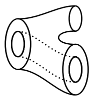
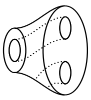
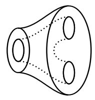
Example 1.19. A circle constellation is for example given by: 

Definition 1.20. The *basic cobordisms for NesCob* are:



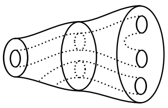
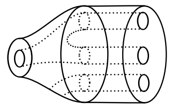
Remark 1.21. Here by  we mean that we take a trouser leg of  and push it down the middle of the other.

Definition 1.22. *Nested basic cobordisms* are defined inductively as follows: We take a basic cobordism or a nested basic cobordism $f : A \to B$ and add a basic cobordism $g : C \to D$ such that the surfaces of f and g do not intersect each other and C lies in $o(A)$ and D lies in $o(B)$.

Example 1.23. Nested basic cobordisms are for example ,  and .

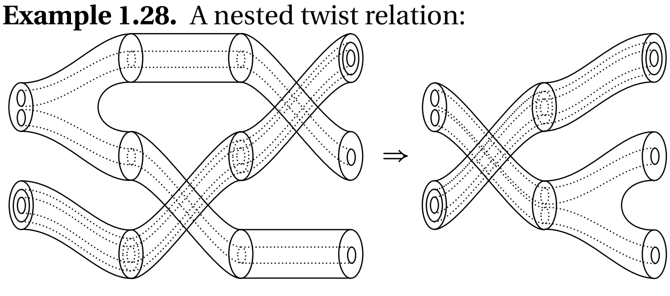
Definition 1.24. The *basic relations for* $NesCob$ are explained in A.3. By *identity relations* we mean the identity relations of Cob together with the identity relations that are only in $NesCob$. *Twist relations* for $NesCob$ are defined analogously.

Definition 1.25. By a *nested identity relation* we mean that an identity relation is applied to a part of a cobordism and the rest remains unchanged.

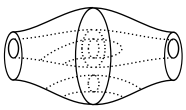
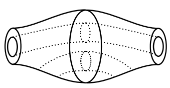
Example 1.26. A nested identity relation:  $\Rightarrow$ 

Definition 1.27. Assume we have cobordism which is the source of a twist relation filled with identity cobordisms. Then a relation obtained by forgetting the identity cobordisms inside, applying the twist relation and then filling with the same identity cobordisms is called a *nested twist relation.*

Example 1.28. A nested twist relation:



Definition 1.29. *Simple nested relations* arise as follows: We apply a basic relation which is not a twist relation to a part of a cobordism in which no further cobordism are nested inside and leave the other part of the cobordism unchanged.

Example 1.30. A simple nested relation:  $\Rightarrow$ 

Definition 1.31. By a *nested basic relation* we mean a nested identity relation, a nested twist relation or a simple nested relation.

NesCob:
NesCob is generated by

- the circle constellations as basic objects,
- the basic cobordisms and the nested basic cobordisms as basic morphisms,
- the basic relations explained in Section A.3 and nested basic relations as basic 2-morphisms,

where the tensor products, horizontal and vertical compositions are given as for *Cob*. For A a circle constellation define id_A as the corresponding nested identities. As for *ColCob* we define $\mathrm{id}_A := \mathrm{id}_{A_{i_1}} \otimes \cdots \otimes \mathrm{id}_{A_{i_k}}$ for A_{i_j} basic objects and $A = A_{i_1} \otimes \cdots \otimes A_{i_k}$ an object in *NesCob*, and 1_f = "apply no relation" for f a morphism. Again, the morphisms are considered modulo the local relations given by the identity relations explained in A.3.

Theorem 1.32. *NesCob is a semistrict monoidal 2-category.*

Proof. Again, *NesCob* is generated in the way required by the Theorem 1.10. The condition a) holds by definition. *NesCob* satisfies the other conditions of the theorem analogously as *Cob* in the proof of Theorem 1.14, because the different concatenations are defined in the same way as for *Cob*. □

1.4 Vect2

Definition 1.33. [CP94] A *monoidal category* is a category $\mathcal{C}$ together with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfying the following conditions

i) There are natural isomorphisms $\alpha_{U,V,W} : U \otimes (V \otimes W) \to (U \otimes V) \otimes W$ such that the pentagon diagram

$$\begin{tikzcd}
& (U\otimes V)\otimes(W\otimes Z) \arrow[dr, "\alpha"] & \\
U\otimes\big(V\otimes(W\otimes Z)\big) \arrow[ur, "\alpha"] \arrow[d, "\mathrm{id}\otimes\alpha"'] & & \big((U\otimes V)\otimes W\big)\otimes Z \\
U\otimes\big((V\otimes W)\otimes Z\big) \arrow[rr, "\alpha"'] & & \big(U\otimes(V\otimes W)\big)\otimes Z \arrow[u, "\alpha\otimes\mathrm{id}"']
\end{tikzcd}$$

commutes for all U, V, W, Z.

ii) There is an identity object I in $\mathcal{C}$ and natural isomorphism $\rho_U : U \otimes I \to U$ and $\lambda_U : I \otimes U \to U$ such that the triangle diagram

$$\begin{tikzcd}
(V \otimes I) \otimes W \arrow[rr, "\alpha_{V,I,W}"] \arrow[dr, "\rho_V \otimes id_W"'] & & V \otimes (I \otimes W) \arrow[dl, "id_V \otimes \lambda_W"] \\
& V \otimes W &
\end{tikzcd}$$

commutes for all V, W.

A monoidal category is called *strict* if the morphisms α, ρ and λ are identity maps.

Later on we consider other monoidal categories, but for now we use the following one:

Vect
Objects: Vector spaces over $\mathbb{C}$.
Morphisms: linear maps

By [Kas95, p. 285] this is a monoidal category with tensor product the tensor product of vector spaces and $\mathbb{C}$ as unit object. Let Vect_c be the associated strictification ([Kas95, p. 288], [KV94, 2.5 Examples]), which we identify with Vect in the following.

Vect2:
Objects: Objects of Vect
Morphisms: Morphisms of Vect
2-Morphisms: The 2-morphisms are the identity Id between the same morphisms and the negative of the identity $-\mathrm{Id}$ between morphisms f and g with $f = -g$.

Theorem 1.34. *Vect2 is a semistrict monoidal 2-category.*

Proof. Since Vect is a category, the objects and the morphisms form a category with the usual composition of linear maps. The horizontal composition of two 2-morphisms $\pm\mathrm{Id} : f \Rightarrow g : A \to B$ and $\pm\mathrm{Id} : f' \Rightarrow g' : B \to C$ is defined by $(\pm 1)(\pm 1)\mathrm{Id} : f \circ f' \Rightarrow g \circ g'$. For this $1_{\mathrm{id}_A} := \mathrm{Id} : \mathrm{id}_A \Rightarrow \mathrm{id}_A$ is an identity. Furthermore, this composition is associative.

We define the vertical composition of two 2-morphisms $\pm\mathrm{Id} : f \Rightarrow g : A \to B$ and $\pm\mathrm{Id} : g \Rightarrow h : A \to B$ by $(\pm 1)(\pm 1)\mathrm{Id} : f \Rightarrow h$. This composition is also associative and $1_f := \mathrm{Id} : f \Rightarrow f$ is the identity. Moreover, we have $1_f \circ 1_g = \mathrm{Id} \circ \mathrm{Id} = \mathrm{Id} = 1_{fg}$ and the "middle four exchange" holds as well. Hence, Vect2 is a 2-category.

Let $I = \mathbb{C}$. We define the tensor product of objects or morphisms as the usual tensor product of objects or morphisms over $\mathbb{C}$, respectively. Since Vect_c is a strict monoidal category, we have h) and l) of Corollary 1.11. Analogously, we have the associativity of the tensor product of objects and morphisms. Since 2-morphisms are only Id or $-\mathrm{Id}$, the associativity of the tensor product for 2- morphisms follows, and we have k). For the same reason we have g), i) and j). Therefore, the assertion follows by Corollary 1.11. $\square$

2 2-Functors

In this section we define monoidal 2-functors and provide a theorem which helps us to consider monoidal 2-functors from Cob, $ColCob$ and $NesCob$ to $Vect2$.

In the previous chapter we already defined the notion of a 2-functor between 2-categories. But now we need the 2-functor also to respect the monoidal structure, which leads to the next definition.

Definition 2.1. ([Lan97]) Let $\mathcal{C}$ and $\mathcal{D}$ be semistrict monoidal 2-categories. A *(strict) monoidal 2-functor* $F:\mathcal{C}\to\mathcal{D}$ consists of

- a 2-functor $F:\mathcal{C}\to\mathcal{D}$
- for any two objects A and B an equality $F(A\otimes B)=FA\otimes FB$
- for any object A and any morphism f, equalities $F(A\otimes f)=FA\otimes Ff$ and $F(f\otimes A)=Ff\otimes FA$
- for any object A and any 2-morphism α, equalities $F(A\otimes\alpha)=FA\otimes F\alpha$ and $F(\alpha\otimes A)=F\alpha\otimes FA$
- $F(I_C)=I_D$
- for any two morphisms f,g equalities $F(\bigotimes_{f,g})=\bigotimes_{F(f),F(g)}$

The following theorem is our main tool for showing that the examples in the next section are monoidal 2-functors.

Theorem 2.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be semistrict monoidal 2-categories such that $\mathcal{C}$ fulfils the conditions of Theorem 1.10 and $\mathcal{D}$ fulfils the conditions of Corollary 1.11. Additionally assume that there are no further equations on object and morphism level in $\mathcal{C}$. Each basic object is not the product of other ones and each basic morphism cannot be expressed by other ones except using identities.*
Then by a specification of $F(A_i)$, $F(f_j)$ and $F(\alpha_k)$ that is compatible with source and target we get a well-defined monoidal 2-functor $F:\mathcal{C}\to\mathcal{D}$, if the equations on 2-morphisms are compatible with F.

Remark 2.3. Here we use the term “equation” where one would usually use “relation” to avoid confusion with the 2-morphisms of the cobordism categories which are also called relations.

Proof. We define $F(I_C):=I_{\mathcal{D}}$, $F(\mathrm{id}_{A_i}):=\mathrm{id}_{F(A_i)}$ and $F(1_f):=1_{F(f)}$.

By assumption we have $A=A_{i_1}\otimes\ldots\otimes A_{i_k}$ for every object A. We define $F(A):=F(A_{i_1})\otimes\ldots\otimes F(A_{i_k})$. Hence, $F(A\otimes B)=F(A)\otimes F(B)$ holds for every objects $A,B\in\mathcal{C}$.

Each morphism f is obtained from the basic morphisms and the id_{A_i} by finitely many tensor products and compositions. We define $F(f)$ to be the same tensor products and compositions of the $F(f_i)$ and $F(\mathrm{id}_{A_i})$. From this we get $F(f \circ g) = F(f) \circ F(g)$ and $F(f \otimes g) = F(f) \otimes F(g)$ for all morphisms $f, g \in \mathcal{C}$, thus in particular $F(A \otimes f) = F(A) \otimes F(f)$ and $F(A \otimes f) = F(A) \otimes F(f)$ for every object $A \in \mathcal{C}$. Since $\mathrm{id}_A = \mathrm{id}_{A_{i_1}} \otimes \ldots \otimes \mathrm{id}_{A_{i_k}}$ and $\mathrm{id}_X \otimes \mathrm{id}_Y = \mathrm{id}_{X \otimes Y}$ holds in $\mathcal{D}$, we have $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

Each 2-morphism α is obtained from the basic 2-morphisms and the 1_f by finitely many tensor products, horizontal and vertical compositions. We define $F(\alpha)$ to be the same tensor products, horizontal and vertical compositions of the $F(\alpha_i)$ and $F(1_f)$. So we get $F(\alpha \circ \beta) = F(\alpha) \circ F(\beta)$, $F(\alpha \bullet \beta) = F(\alpha) \bullet F(\beta)$ and $F(\alpha \otimes \beta) = F(\alpha) \otimes F(\beta)$, thus in particular $F(A \otimes \alpha) = FA \otimes F\alpha$ and $F(\alpha \otimes A) = F\alpha \otimes FA$.

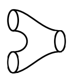
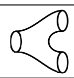
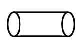
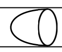
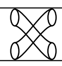
Since by assumption the condition h) of Corollary 1.11 is satisfied in $\mathcal{D}$, we get as in the proof of 1.10 that $\otimes_{f,g} = 1_{f \otimes g}$ also holds in $\mathcal{D}$. Thus $F(\otimes_{f,g}) = F(1_{f \otimes g}) = 1_{F(f \otimes g)} = 1_{F(f) \otimes F(g)} = \otimes_{F(f),F(g)}$.

Altogether, F satisfies the conditions for being a monoidal 2-functor.

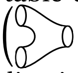
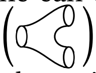
The compatibility of source and target with F is preserved under compositions and tensor products. Since every equation on objects in $\mathcal{C}$ also holds in $\mathcal{D}$ and the basic objects cannot be expressed by other ones, F is well-defined on objects. The conditions b) and c) of Theorem 1.10 are satisfied in every 2-category, thus also in $\mathcal{D}$. Thus, every equation on morphisms in $\mathcal{C}$ also holds in $\mathcal{D}$. Since the basic morphisms cannot be expressed by each other up to identities and identities are compatible with F, we have that F is well-defined on morphisms. By assumption the equations on 2-morphisms are compatible with F, so F is well-defined on 2-morphisms. □

2.1 2-Functor for Cob

With the theorem above we now want to construct a special monoidal 2-functor from Cob to $Vect2$. We use the following table to define the 2-functor on morphisms.

	$1 \otimes 1 \mapsto 1$ $x \otimes 1 \mapsto x$ $1 \otimes x \mapsto x$ $x \otimes x \mapsto 0$
	$1 \mapsto x \otimes 1 + 1 \otimes x$ $x \mapsto x \otimes x$
	$1 \mapsto 1$ $x \mapsto x$
	$1 \mapsto 1$
	$1 \mapsto 0$ $x \mapsto 1$
	$a \otimes b \mapsto b \otimes a$

Remark 2.4. If we consider all the 2-morphisms in Cob as relations, divide them out and call the result Cob', then we get: A monoidal functor Z from Cob' to Vect is nothing else than a Frobenius algebra, i.e. a finite-dimensional commutative algebra with a non-degenerate associative pairing [Abr96].

What we want to consider in this section is similar to the following special case for the monoidal functor Z: Let $Z(O) := \mathbb{C}[x]/(x^2)$ for O the circle in Cob'. We define Z on cobordisms via the table above. One can check that this defines a monoidal functor. Furthermore, call $Z\Big($$\Big) =: m$, $Z\Big($$\Big) =: \Delta$, $Z\big($$\big) =: \epsilon$ and $Z\big($$\big) =: \eta$ for multiplication, comultiplication, unit and counit. Then one can compute that the Frobenius relation $(\mathrm{id}\otimes\Delta)\circ(m\otimes\mathrm{id}) = m\circ\Delta = (\Delta\otimes\mathrm{id})\circ(\mathrm{id}\otimes m)$ holds. Thus $(\mathbb{C}[x]/(x^2), m, \epsilon, \Delta, \eta)$ is a Frobenius algebra by [Koc04, 2.3.24]. However, at the moment we divide out fewer relations and consider Cob instead of Cob'.

Theorem 2.5. *Let $V = \mathbb{C}[x]/(x^2)$ and let O be the circle in Cob. Then a well-defined monoidal 2-functor $F = F_{Cob} : Cob \to Vect2$ is given by*

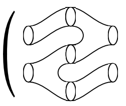
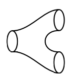
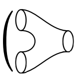
$$\begin{aligned} F(O) &= V, \\ F(\textit{basic cobordism}) &= \textit{map from table above}, \\ F(\textit{basic relation}) &= \mathrm{Id}. \end{aligned}$$

Proof. By Section 1 Cob satisfies the conditions of Theorem 1.10 and Vect2 the ones of Corollary 1.11.

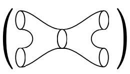
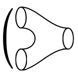
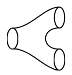
Since O is the only basic object, it cannot be expressed by other basic objects. There are no equations on objects except the ones given by the conditions of a semistrict monoidal 2-category. The basic cobordisms cannot be expressed by others except by Identity relations. Since we only divided out the identity relations, there are no further equations on the morphisms than given by Theorem 1.10.

Since all basic relations are mapped to Id, the equations on 2-morphisms are compatible with F.

The source and target of basic cobordisms are compatible with F by definition. For $\alpha : f \Rightarrow f'$ a basic relation one can compute case by case that $F(f) = F(f')$. For example:

$F\Big($$\Big) = \Big(\mathrm{id}\otimes F\Big($$\Big)\Big)\circ\Big(F\Big($$\Big)\otimes\mathrm{id}\Big)$

$$= \begin{pmatrix} 1\otimes 1 & \mapsto & 1\otimes 1\otimes x + 1\otimes x\otimes 1 & \mapsto & 1\otimes x + x\otimes 1 \\ 1\otimes x & \mapsto & 1\otimes x\otimes x & \mapsto & x\otimes x \\ x\otimes 1 & \mapsto & x\otimes 1\otimes x + x\otimes x\otimes 1 & \mapsto & x\otimes x \\ x\otimes x & \mapsto & x\otimes x\otimes x & \mapsto & 0 \end{pmatrix}$$

$F\Big($$\Big) = F\Big($$\Big)\circ F\Big($$\Big)$

$$= \begin{pmatrix} 1\otimes 1 & \mapsto & 1 & \mapsto & 1\otimes x + x\otimes 1 \\ 1\otimes x & \mapsto & x & \mapsto & x\otimes x \\ x\otimes 1 & \mapsto & x & \mapsto & x\otimes x \\ x\otimes x & \mapsto & 0 & \mapsto & 0 \end{pmatrix}$$

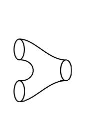
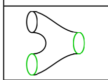
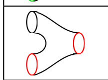
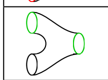
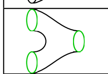
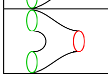
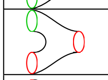
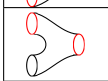
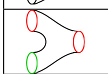
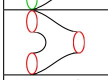
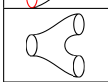
The other cases can be checked easily in a similar way. Thus, we have $\mathrm{Id} = F(\alpha) : F(f) \Rightarrow F(f')$ and the source and target of basic 2-morphisms are compatible with F. Therefore, the assertion follows by Theorem 2.2. $\square$

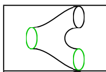
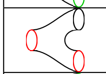
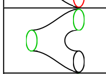
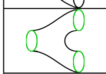
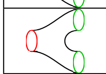
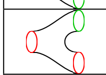
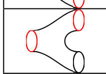
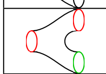
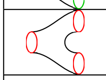
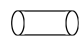
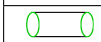
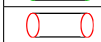
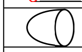
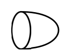
2.2 2-Functor for ColCob

Similar to the last section we now define a certain monoidal 2-functor from $ColCob$ to $Vect2$.

Theorem 2.6. *Let $V = \mathbb{C}[x]/(x^2)$ and let B be the black circle in $ColCob$, R the red one and G the green one. Then a well-defined monoidal 2-functor $F = F_{ColCob} : ColCob \to Vect2$ is given by*

$$\begin{aligned} F(S) = V, \quad F(G) &= \mathbb{C}, \quad F(R) = 0, \\ F(\textit{basic cobordism}) &= \textit{map from table below}, \\ F(\textit{basic relation}) &= \mathrm{Id}. \end{aligned}$$

	$1\otimes 1 \mapsto 1$ $x\otimes 1 \mapsto x$ $1\otimes x \mapsto x$ $x\otimes x \mapsto 0$
	$1\otimes 1 \mapsto 1$ $x\otimes 1 \mapsto 0$
	$1\otimes 0 \mapsto 0$ $x\otimes 0 \mapsto 0$
	$1\otimes 1 \mapsto 1$ $1\otimes x \mapsto 0$
	$1\otimes 1 \mapsto 1$
	$1\otimes 1 \mapsto 0$
	$1\otimes 0 \mapsto 0$
	$0\otimes 1 \mapsto 0$ $0\otimes x \mapsto 0$
	$0\otimes 1 \mapsto 0$
	$0\otimes 0 \mapsto 0$
	$1 \mapsto x\otimes 1 + 1\otimes x$ $x \mapsto x\otimes x$

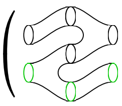
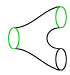
	$1 \mapsto x\otimes 1$
	$0 \mapsto 0\otimes 0$
	$1 \mapsto 1\otimes x$
	$1 \mapsto 0\otimes 0$
	$0 \mapsto 0\otimes 0$
	$0 \mapsto 0\otimes 0$
	$0 \mapsto 0\otimes 0$
	$0 \mapsto 0\otimes 0$
	$0 \mapsto 0\otimes 0$
	$1 \mapsto 1$ $x \mapsto x$
	$1 \mapsto 1$
	$0 \mapsto 0$
	$1 \mapsto 1$
	$1 \mapsto 0$ $x \mapsto 1$
Twists	$a\otimes b \mapsto b\otimes a$

Proof. By Section 1 $ColCob$ satisfies the conditions of Theorem 1.10 and Vect2 the ones of Corollary 1.11.

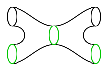
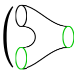
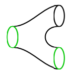
The basic objects B, R and G are not the tensor product of each other, since we cannot change their colours by tensoring them together. There are no equations on objects except the ones given by the conditions of a semistrict monoidal 2-category. The basic cobordisms cannot be expressed by others except by identity relations. Since we only divided out the identity relations, there are no further equations on the morphisms than given by Theorem 1.10.

Since all basic relations are mapped to Id, the equations on 2-morphisms are compatible with F.

The source and target of basic cobordisms are by definition compatible with F. For $\alpha : f \Rightarrow f'$ a basic relation one can compute $F(f) = F(f')$. We present one example and omit the rest:

$F\Big($  $\Big) = \Big(\mathrm{id}_S \otimes F\big($  $\big)\Big) \circ \Big(F\big($  $\big) \otimes \mathrm{id}_G\Big)$

$$= \begin{pmatrix} 1\otimes 1 & \mapsto & 1\otimes x\otimes 1 & \mapsto & x\otimes 1 \\ x\otimes 1 & \mapsto & x\otimes x\otimes 1 & \mapsto & 0 \end{pmatrix}$$

$F\Big($  $\Big) = F\big($  $\big) \circ F\big($  $\big)$

$$= \begin{pmatrix} 1\otimes 1 & \mapsto & 1 & \mapsto & x\otimes 1 \\ x\otimes 1 & \mapsto & 0 & \mapsto & 0 \end{pmatrix}$$

Thus, we have $\mathrm{Id} = F(\alpha) : F(f) \Rightarrow F(f')$ and the source and target of basic 2-morphisms are compatible with F.

Therefore, the assertion follows by Theorem 2.2. □

2.3 Relationship between F_{Cob} and F_{ColCob}

Forgetting the colours defines a monoidal 2-functor $T : ColCob \to Cob$.

For A an object in $ColCob$ we define $i_A : F_{ColCob}(A) \to F_{Cob}(T(A))$ by

$$\begin{aligned} i_B : \mathbb{C}[x]/(x^2) &\xrightarrow{\mathrm{id}} \mathbb{C}[x]/(x^2) \\ i_G : \mathbb{C} &\hookrightarrow \mathbb{C}[x]/(x^2) \\ 1 &\mapsto 1 \\ i_R : \{0\} &\hookrightarrow \mathbb{C}[x]/(x^2) \\ \text{and } i_{X\otimes Y} &= i_X \otimes i_Y \end{aligned}$$

for B, G, R black, green, red circles, respectively, and X, Y arbitrary objects in $ColCob$.

For A an object in $ColCob$ we define $\pi_A : F_{Cob}(T(A)) \to F_{ColCob}(A)$ by

$$\begin{aligned} \pi_B : \mathbb{C}[x]/(x^2) &\xrightarrow{\mathrm{id}} \mathbb{C}[x]/(x^2) \\ \pi_G : \mathbb{C}[x]/(x^2) &\to \mathbb{C} \\ 1 &\mapsto 1 \\ x &\mapsto 0 \\ \pi_R : \mathbb{C}[x]/(x^2) &\xrightarrow{0} \{0\} \\ \text{and } \pi_{X\otimes Y} &= \pi_X \otimes \pi_Y \end{aligned}$$

for B, G, R, X, Y as above.

Proposition 2.7. *For $f : A \to A'$ in $ColCob$ the following diagram commutes:*

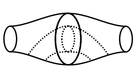
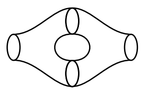
$$\begin{tikzcd} F_{ColCob}(A) \arrow[r, "i_A"] \arrow[d, "F_{ColCob}(f)"'] & F_{Cob}(T(A)) \arrow[d, "F_{Cob}(T(f))"] \\ F_{ColCob}(A') & F_{Cob}(T(A')) \arrow[l, "\pi_{A'}"'] \end{tikzcd}$$

Proof. This holds, since $F_{ColCob}(f) = i_A \circ F_{Cob}(T(f)) \circ \pi_{A'}$ is true for the basic cobordisms, as one can see in the tables. Consequently, it extends to all cobordisms. $\square$

2.4 2-Functor for NesCob

In this section we define a certain monoidal 2-functor from $NesCob$ to $Vect2$, which is similar to Sections 2.1 and 2.2 but more laborious.

Definition 2.8. For the circle constellations of $NesCob$ we define the *degree of nestedness* of the occurring circles: deg(circle) is defined to be the number of circles it contains. (Note: Each circle contains itself.)
For M a basic object of degree n let $M_{<n} := M \setminus o(M)$ be the inner circles.

By *torus relation* we mean the relation (A.3.7)  $\Leftrightarrow$  .

Theorem 2.9. *Let $V = \mathbb{C}[x]/(x^2)$. A well-defined monoidal 2-functor*

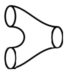
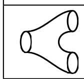
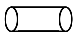
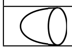
$$F = F_{NesCob} : NesCob \to Vect2$$

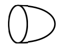
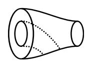
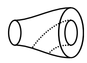
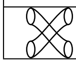
is given by the following:
Let M be a basic object in $NesCob$. For $\deg(M) = 1$ *define $F(M) = V$ and for* $\deg(M) = n > 1$ *define $F(M) = V \otimes F(M_{<n})$ keeping in mind $F(A \otimes B) = F(A) \otimes F(B)$.*

$$\begin{aligned} F(\textit{basic cobordism}) &= \textit{map from table below}, \\ F(\textit{nested basic cobordism}) &= f, \end{aligned}$$

where f is given by: If a basic cobordism b connects certain circles, then f restricts to $F(b)$ on the associated V's.

$$\begin{aligned} F(\textit{torus relation}) &= -\mathrm{Id}, \\ F(\textit{other basic relation}) &= \mathrm{Id}, \\ F(\textit{nested basic relation}) &= F(\textit{associated basic relation}) \end{aligned}$$

	$1\otimes 1\mapsto 1$ $x\otimes 1\mapsto x$ $1\otimes x\mapsto x$ $x\otimes x\mapsto 0$
	$1\mapsto 1\otimes x+x\otimes 1$ $x\mapsto x\otimes x$
	$1\mapsto 1$ $x\mapsto x$
	$1\mapsto 1$

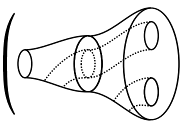
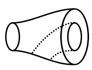
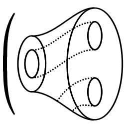
	$1\mapsto 0$ $x\mapsto 1$
	$1\otimes 1\mapsto 1$ $x\otimes 1\mapsto x$ $1\otimes x\mapsto -x$ $x\otimes x\mapsto 0$
	$1\mapsto 1\otimes x-x\otimes 1$ $x\mapsto x\otimes x$
	$a\otimes b\mapsto b\otimes a$

Proof. By section 1 $NesCob$ satisfies the conditions of Theorem 1.10 and Vect2 the ones of Corollary 1.11.

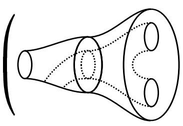
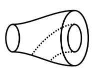
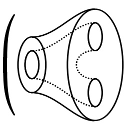
The tensor product only allows us to put circle constellations next to each other and not into each other. This implies that the basic objects given by circle constellations are not the tensor product of each other. For the same reason, the basic cobordisms and nested basic cobordisms are not represented by others except by identity relations. Again, there are no equations on objects except the ones given by the conditions of a semistrict monoidal 2-category. Since we only divided out the identity relations and the nested identity relations, there are no further equations on the morphisms than given by Theorem 1.10.

The torus relation and the nested torus relations cannot be expressed by other basic relations and nested basic relations. In addition, they cannot be expressed by each other since they only work on the inner level of morphisms. If one of the remaining basic relations or nested basic relations can be expressed by other ones, then this is compatible with F since all the remaining basic relations and nested basic relations are mapped to Id.

The source and target of basic cobordisms are by definition compatible with F, thus this also holds for nested basic cobordisms. For $\alpha : f \Rightarrow f'$ a basic relation which is not a torus relation one can compute that $F(f) = F(f')$. We present one example and omit the rest:

$F\Big($  $\Big) = F\Big($  $\Big) \circ F\Big($  $\Big)$

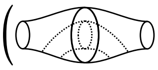
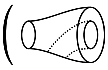
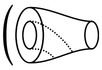
$$= \begin{pmatrix} 1 & \mapsto & 1\otimes x - x\otimes 1 & \mapsto & 1\otimes x\otimes x - x\otimes x\otimes 1 - x\otimes 1\otimes x \\ x & \mapsto & x\otimes x & \mapsto & x\otimes x\otimes x \end{pmatrix}$$

$F\Big($  $\Big) = F\Big($  $\Big) \circ F\Big($  $\Big)$

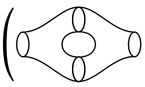
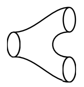
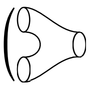
$$= \begin{pmatrix} 1 & \mapsto & 1\otimes x - x\otimes 1 & \mapsto & 1\otimes x\otimes x - x\otimes 1\otimes x - x\otimes x\otimes 1 \\ x & \mapsto & x\otimes x & \mapsto & x\otimes x\otimes x \end{pmatrix}$$

Thus, in this case we have $\mathrm{Id} = F(\alpha) : F(f) \Rightarrow F(f')$.

For the torus relation we can compute $F(f) = -F(f')$:

$F\Big($  $\Big) = F\Big($  $\Big) \circ F\Big($  $\Big)$

$$= \begin{pmatrix} 1 & \mapsto & 1 \otimes x - x \otimes 1 & \mapsto & -2x \\ x & \mapsto & x \otimes x & \mapsto & 0 \end{pmatrix}$$

$F\Big($  $\Big) = F\Big($  $\Big) \circ F\Big($  $\Big)$

$$= \begin{pmatrix} 1 & \mapsto & 1 \otimes x + x \otimes 1 & \mapsto & 2x \\ x & \mapsto & x \otimes x & \mapsto & 0 \end{pmatrix}$$

Thus we get $-\mathrm{Id} = F(\alpha) : F(f) \Rightarrow F(f')$ for $\alpha : f \Rightarrow f'$ the torus relation.

This extends to nested basic relations, so they are also compatible with F.

Therefore, the assertion follows by Theorem 2.2. □

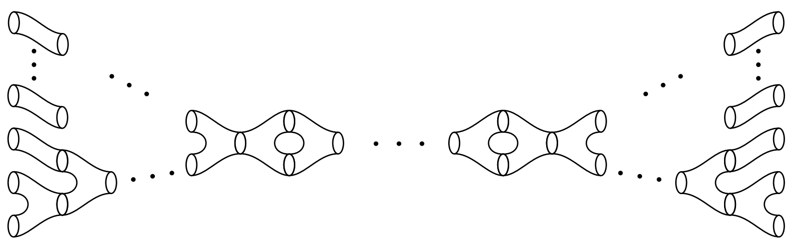
3 Normal forms

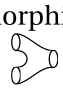
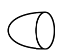
In this section we show that the connected morphisms in Cob, $ColCob$ and $NesCob$ can be transformed to a certain normal form by applying 2-morphisms.

Definition 3.1. A morphism in Cob, $ColCob$ or $NesCob$ is called *connected*, if the topological surface, we get by forgetting the glueing points, is path connected.

3.1 Normal form for Cob

Theorem 3.2 (normal form for Cob)**.**
By applying finitely many 2-morphisms any connected morphism in Cob *can be transformed into the following normal form:*



Proof. (analogous to [Koc04])
Step 1: A connected morphism which contains no twist can be transformed into the above normal form by 2-morphisms:
Let m be the number of in-boundaries in the morphisms, n the number of out-boundaries, a the number of , b the number of , p the number of  and q the

number of . If there are only and to the left of a , then the in-part of the morphism can be transformed into normal form by (A.1.5). The analogous statement holds for with the reflected counterpart of the 2-morphism (A.1.5) for the out-part.

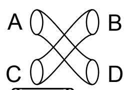
Now, assume there is a with not only and to the left of it. We move this to the left by (A.1.2), until it adjoins on the left to something other than . We distinguish the following cases:

- adjoins. We use (A.1.4). So p and b decrease by 1.
- adjoins:
    - adjoins sloping as in . We use (A.1.7) to move to the left of . In this case, the numbers stay the same but now there are fewer to the left of .
    - adjoins as in . We leave them that way.
- adjoins and at the left of it there is something else than and .
    - the left one of the two adjoins directly to a as in . We use (A.1.5) on the right part of what is build by the three basic cobordisms and identities, and after that we use (A.1.7) on the left part to move a past the .
    - the left one of the is not in this situation. We go on with this .

Now we iterate until all build a , which we call hole, or at the left of them there are only or . After that we analogously move the to the right.

Note: If there is none or only one in-boundary or out-boundary, then we need no or for the in-part or out-part of the normal form, respectively, but or or .

Step 2: killing the twists:

Let T be a twist and we label its boundaries as in 

. Now we use the identity relations (A.1.1) - (A.1.3) until there are, if any, only parallel to T. Since the morphism is connected, some of A - D have to be joint by other parts of the morphism. We distinguish the following cases:

- B and D are joint. We consider the part of the morphism (without T) which is connected to B and D. This part contains fewer twists than the morphism we started with, hence by induction it can be transformed into normal form. Therefore, the in-part of that new normal form adjoins to B and D. Now we use (A.1.5) to move a such that it adjoins T in the way that we can use (A.1.6). With (A.1.6) we kill the twist.

- A and C are joint. We do the same as above with the reflected counterparts of the 2-morphisms.

- A and B are joint. We delete T and all basic morphisms parallel to it. Then by induction the remaining parts can be transformed into normal form. Since A is not connected to C and B is not connected to D (we already considered this cases), the part connected to A and B can be transformed into normal form such that the basic morphisms around T are of the form . Now we apply the 2-morphisms (A.1.6) (reflected), (A.1.10), (A.1.7) and (A.1.6) (reflected) (as in [Koc04, 1.4.38]), which kills the twist T.

- C and D are joint. This is similar to the situation that A and B are connected, but we have to use that with (A.1.5) we can change the normal form such that the outer and are at the top and not at the bottom.

□

Remark 3.3. The normal form is uniquely determined by the morphism we started with: Again, let m be the number of in-boundaries in the morphisms, n the number of out--boundaries, a the number of , b the number of , p the number of and q the number of . Then, the in-part and the out-part are determined by m or n, respectively. The numbers a, b, p and q are not changed by killing the twists since this is true in every relation appearing in Step 2.

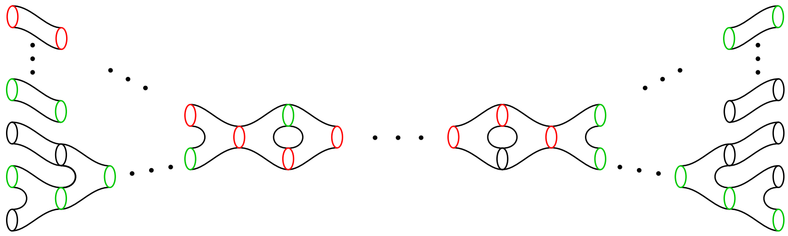
Independent of the shape of the in- and out-part of the normal form there are

$$b-(m-1)-p = a-(n-1)-q = \frac{1}{2}((b+a)-(m+n-2)-(p+q))$$

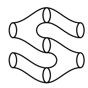
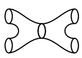
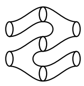
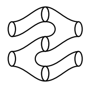
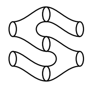
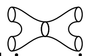
holes. This holds, because for $m \geq 1$ we use $m-1$ for the in-part and p cancel with . The remaining ones form half of a hole. Since at the beginning all the parts formed a morphism, we cannot have a single without a partner forming a hole with it. If $m = 0$, then we have that the number of the which are not in the in-part or killed by is equal to $b-(p-1) = b-m-p+1 = b-(m-1)-p$, since one remains. This explains the first part of the equation. The second part arises by the same argument for the , and the third part is the mean of the first two parts.

3.2 Normal form for ColCob

Theorem 3.4 (normal form for ColCob)**.**
By applying finitely many 2-morphisms any connected morphism in ColCob can be transformed into the following normal form

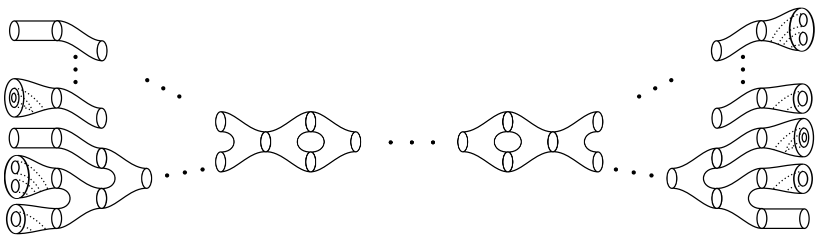


where the colours may differ depending on the morphism we started with.

Proof. We use the same procedure as for Cob. To do this, we have to check that the 2-morphisms we used also exist in $ColCob$ in every possible colouring: In $ColCob$ we have the relations (A.1.1) - (A.1.11) except (A.1.7) in all colourings and reflections for which the basic cobordisms exist in these colourings. The 2-morphism (A.1.7)  $\Leftrightarrow$  $\Leftrightarrow$  does not exist in every colouring. But if  or  exists in a certain colouring, then there is a 2-morphism to  in the same colours, see A.2. Since the procedure for Cob only uses the 2-morphisms in that direction, it also works for $ColCob$. □

3.3 Normal form for NesCob

Theorem 3.5 (normal form for NesCob)**.**
By applying finitely many 2-morphisms any connected morphism in NesCob can be transformed into the following normal form:



Proof. In this proof we do not differentiate between relations and the associated nested relations.

Let the inner hull be the morphism level which contains no further cobordism.

If there is no morphism level except the inner hull, we use the procedure for Cob.

Otherwise, let l_1 the number of or , k_1 the number of or , p_1 the number of , n_1 the number of or and m_1 the number of boundary circles of the inner hull.

Since the morphism is connected, all basic cobordisms , , , , and all boundary circles of the inner hull have to be directly or indirectly connected to or .

We consider a or and detect the basic cobordism or boundary circle it is first connected to. We only examine the cases for since the cases for are just the reflections of the ones for .

If is connected to one of the above, we use (A.3.1) - (A.3.5) until is directly connected to it. (This is possible, since all morphisms are build by finitely many basic morphisms). Now we distinguish the following cases:

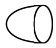
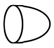
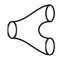
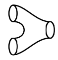
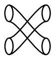
is directly connected to

- :
  We use (A.3.8). l_1 and n_1 decrease by 1.
- :
  We use (A.3.6). k_1 decreases by 1 and n_1 increases by 1.
- :
  We use (A.3.9). k_1 decreases by 1 and n_1 increases by 1.
- :
  We use (A.3.10). p_1 decreases by 1 and n_1 stays the same.
- a boundary circle:
  So this part is already in normal form and is not changed again. We decrease m_1 and n_1 by 1.
- :
  We use (A.3.7). n_1 decreases by 2.

We iterate with the next and then go on with . This way, the number of , , , , and involved boundary circles decreases until all are vanished, since they all are connected to or and the 2-morphisms which we apply to other parts do not change that. In contrast to that, n_1 increases as long as there are or left, but only finitely many times. Then it decreases by 2 when (A.3.7) is applied. Since everything is connected, after finitely many 2-morphisms there is nothing left of the inner hull.

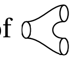
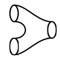
Now, we have a new inner hull. We use the same procedure again until it is also vanished and iterate. □

Remark 3.6. Again, the morphism we started with completely determines the normal form.

Indeed, let l_r the number of  or , k_r the number of  or , p_r the number of , n_r the number of  or  and m_1 the number of boundary circles of the rth inner hull. In each step, $\frac{1}{2}(n_r - m_r - l_r + k_r)$ holes were added to the former $(r+1)$st inner hull, since two of the ones counted by n_r cause a hole and the ones counted by m_r and l_r decrease n_r by 1 whereas the ones counted by k_r increase n_r by 1.

Altogether, if the morphism has N levels, the normal form we get consists of

$$\frac{a_{N-1}}{2} + \frac{1}{2}(k_N - (m_N - 2) - l_N)$$

holes and the in-part and the out-part. Here a_{N-1} is the number of  and  coming from the $(N-1)$st level, so $a_{N-1} = \sum_{i=1}^{N-1}(n_i - m_i - k_i + l_i)$, and the last part of the equation is the formula for the number of holes in the normal form for Cob.

4 Symmetric monoidal categories

Since it is complicated to show that Cob, $ColCob$ and $NesCob$ are semistrict symmetric monoidal 2-categories, we only show that we have a symmetry if we see the 2-morphisms as relations and divide them out.

Definition 4.1. For $\mathcal{C}$ a to category we define $\mathcal{C}'$ to be the category that arises from $\mathcal{C}$ by dividing out all 2-morphisms.

In particular, we now have Cob', $ColCob'$ and $NesCob'$. On their cobordisms the glueing points are not visible any more.

Remark 4.2. We can rewrite the definition of a strict monoidal category as follows, cf. [Koc04]:

A strict monoidal category consists of a category $\mathcal{C}$ together with:

1) An object $I \in \mathcal{C}$ called the unit.

2) For any two objects $A, B \in \mathcal{C}$ an object $A \otimes B$ in $\mathcal{C}$.

3) For any morphisms $f : A \to A'$ and $g : B \to B'$ a morphism $f \otimes g : A \otimes B \to A' \otimes B'$.

This data is subject to the following conditions.

i) For A any object and for f any morphism in $\mathcal{C}$ we have $A \otimes I = I \otimes A = A$ and $f \otimes \mathrm{id}_I = \mathrm{id}_I \otimes f = f$.

ii) For all objects $A, B, C \in \mathcal{C}$ we have $A \otimes (B \otimes C) = (A \otimes B) \otimes C$.

iii) For all morphisms $f, g, h \in \mathcal{C}$ we have $f \otimes (g \otimes h) = (f \otimes g) \otimes h$.

iv) For any morphisms $f : A \to A'$, $f' : A' \to A''$, $g : B \to B'$ and $g' : B' \to B''$ we have $(f \otimes g) \circ (f' \otimes g') = (f \circ f') \otimes (g \circ g')$.

v) For all objects $A, B \in \mathcal{C}$ we have $id_{A \otimes B} = id_A \otimes id_B$.

Proposition 4.3. *If $\mathcal{C}$ satisfies the conditions of Theorem 1.10, then $\mathcal{C}'$ is a strict monoidal category.*

Proof. By dividing out the 2-morphisms there are only more equations satisfied, not fewer. Since a category $\mathcal{C}$ which satisfies the conditions of Theorem 1.10 is a 2-category, the objects and the morphisms of $\mathcal{C}$ form a category; in particular $\mathcal{C}'$ is a category.

Now we check the conditions of Remark 4.1: By assumption of Theorem 1.10 we have 1) such that i) holds by Theorem 1.10 f). Since $\mathcal{C}$ is a semistrict monoidal 2-category, we have 2). 3) is also satisfied because of the assumption of Theorem 1.10.

iv) is exactly the condition h) of Theorem 1.10 and ii) and iii) follow from k). Furthermore, as shown in the proof of Theorem 1.10, $\mathcal{C}$ satisfies also v) and so $\mathcal{C}'$ satisfies it as well. □

Corollary 4.4. *Cob', $ColCob'$ and $NesCob'$ are strict monoidal categories.*

Proof. As shown in Section 1 Cob, $ColCob$ and $NesCob$ satisfy the conditions of Theorem 1.10. □

Definition 4.5. ([Koc04])
A *symmetric monoidal category* is a strict monoidal category $(\mathcal{C}, \otimes, I)$ together with a map

$$R_{A,B} : A \otimes B \to B \otimes A$$

for each pair of objects A, B subject to the following three axioms:

a) For every pair of morphisms $f : A \to A'$, $g : B \to B'$ the following diagram commutes:

$$\begin{tikzcd} A \otimes B \arrow[r, "f \otimes g"] \arrow[d, "R_{A,B}"'] & A' \otimes B' \arrow[d, "R_{A',B'}"] \\ B \otimes A \arrow[r, "g \otimes f"'] & B' \otimes A' \end{tikzcd}$$

b) For every triple of objects A, B, C, the following diagrams commute:

$$\begin{tikzcd} A \otimes B \otimes C \arrow[rr, "R_{A,B \otimes C}"] \arrow[dr, "R_{A,B} \otimes C"'] & & B \otimes C \otimes A \\ & B \otimes A \otimes C \arrow[ur, "B \otimes R_{A,C}"'] & \end{tikzcd}$$

$$\begin{tikzcd}
A\otimes B\otimes C \arrow[rr, "R_{A\otimes B,C}"] \arrow[dr, "A\otimes R_{B,C}"'] & & C\otimes A\otimes B \\
& A\otimes C\otimes B \arrow[ur, "R_{A,C}\otimes B"'] &
\end{tikzcd}$$

c) For every pair of objects A, B we have $R_{A,B}R_{B,A} = \mathrm{id}_{A\otimes B}$.

4.1 Symmetry for Cob'

We define $R_{A,B} : A\otimes B \to B\otimes A$ inductively:
For $A = I$ or $B = I$ let $R_{A,B} = \mathrm{id}$. Now let K_1, K_2 be objects in Cob' consisting of one circle. Then we define R_{K_1,K_2} to be the twist. For K a circle and A, B objects in Cob' we define

$$R_{A\otimes K,B} = (A\otimes R_{K,B})\circ(R_{A,B}\otimes K) \tag{4.1}$$

$$R_{A,K\otimes B} = (R_{A,K}\otimes B)\circ(K\otimes R_{A,B}). \tag{4.2}$$

Lemma 4.6. *$R_{A,B}$ defined as above is well-defined and we have*

$$R_{A,B\otimes C} = (R_{A,B}\otimes C)\circ(B\otimes R_{A,C}) \text{ and} \tag{4.3}$$

$$R_{A\otimes B,C} = (A\otimes R_{B,C})\circ(R_{A,C}\otimes B). \tag{4.4}$$

Proof. We compute for circles K, L that $R_{A\otimes K,L\otimes B}$ gives the same equation independent of which way of computing we choose:

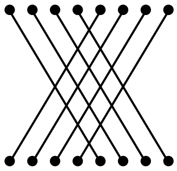
$$\begin{aligned}
R_{A\otimes K,L\otimes B} &\overset{(4.1)}{=} (A\otimes R_{K,L\otimes B})\circ(R_{A,L\otimes B}\otimes K)\\
&\overset{(4.2)}{=} (A\otimes R_{K,L}\otimes B)\circ(A\otimes L\otimes R_{K,B})\circ(R_{A,L}\otimes B\otimes K)\circ(L\otimes R_{A,B}\otimes K)\\
&= (A\otimes R_{K,L}\otimes B)\circ(R_{A,L}\otimes R_{K,B})\circ(L\otimes R_{A,B}\otimes K)
\end{aligned}$$

$$\begin{aligned}
R_{A\otimes K,L\otimes B} &\overset{(4.2)}{=} (R_{A\otimes K,L}\otimes B)\circ(L\otimes R_{A\otimes K,B})\\
&\overset{(4.1)}{=} (A\otimes R_{K,L}\otimes B)\circ(R_{A,L}\otimes K\otimes B)\circ(L\otimes A\otimes R_{K,B})\circ(L\otimes R_{A,B}\otimes K)\\
&= (A\otimes R_{K,L}\otimes B)\circ(R_{A,L}\otimes R_{K,B})\circ(L\otimes R_{A,B}\otimes K)
\end{aligned}$$

For (4.4) we write $B = K_1\otimes\ldots\otimes K_n$. Then the assertion follows by induction, because

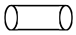
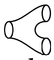
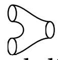
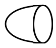
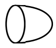
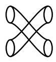
$$\begin{aligned}
&R_{A\otimes K_1\otimes\ldots\otimes K_{n-1}\otimes K_n,C}\\
&\overset{(4.1)}{=} (A\otimes K_1\otimes\ldots\otimes K_{n-1}\otimes R_{K_n,C})\circ(R_{A\otimes K_1\otimes\ldots K_{n-1},C}\otimes K_n)\\
&\overset{IH}{=} (A\otimes K_1\otimes\ldots\otimes K_{n-1}\otimes R_{K_n,C})\circ(A\otimes R_{K_1\otimes\ldots K_{n-1},C}\otimes K_n)\circ(R_{A,C}\otimes K_1\otimes\ldots\otimes K_n)\\
&\overset{(4.1)}{=} (A\otimes R_{K_1\otimes\ldots\otimes K_n,C})\circ(R_{A,C}\otimes K_1\otimes\ldots\otimes K_n).
\end{aligned}$$

Analogously one can show (4.3). □

Since we particularly divided out the identity relations, $R_{A,B}$ has the following form:


Theorem 4.7. *$(Cob', \otimes, I, R_{-,-})$ is a symmetric monoidal category.*

This theorem and its proof are well-known (e.g. [Koc04]), but since the proofs for $ColCob'$ and $NesCob'$ work analogously, we mention it here:

Proof. We check the conditions of Definition 4.5: By Lemma 4.6 the morphism $R_{A,B}$ exists and satisfies the condition b). The condition c) holds since we divided out the 2-isomorphism (A.1.8). Finally, the condition a) follows from the twist relations (A.1.8) - (A.1.11): Since we divided out the identity relations, we can represent each morphism such that there are only  parallel to each , , ,  or . Then we can use the twist relations to move $R_{A,B}$ past each such line. Note that we never are in the situation of (A.1.6) since we consider $f \otimes g$ or $g \otimes f$. □

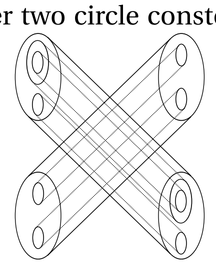
4.2 Symmetry for $ColCob'$

In the same way as for Cob' we can inductively define the twist map in $ColCob'$. The only difference is that this time the circles are coloured. This is possible since in $ColCob$ the twist map exists in all colourings. If we define $R_{A,B}$ in the same manner as before, the statement analogous to Lemma 4.6 holds since the colouring has no impact on the calculations.

Theorem 4.8. *$(ColCob', \otimes, I, R_{-,-})$ is a symmetric monoidal category.*

Proof. This holds similar to Theorem 4.7. By the statement analogous to Lemma 4.6 we have the condition b) of Definition 4.5. The twist relations (A.1.8) - (A.1.11) hold in every possible colouring, so the conditions a) and c) follow similar to the theorem for Cob'. □

4.3 Symmetry for $NesCob'$

Here we need, in addition to the twist maps appearing in the definition for Cob', twist maps for nested circles. Consider two circle constellations A and B. Then we have for example the following twist map: 

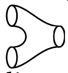
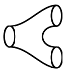
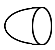
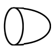
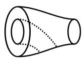
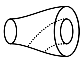
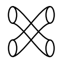
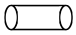
It consists of a normal twist map on $o(A)$ and $o(B)$ and nested identities inside for all circles nested in A or B, respectively. Thus, it is in particular a morphism in $NesCob$ and so in $NesCob'$.

For arbitrary objects in $NesCob'$, i.e. tensor products of circle constellations, we define the twist map inductively in the same way as for Cob'. Again, we have the statement analogous to Lemma 4.6.

Theorem 4.9. $(NesCob', \otimes, I, R_{-,-})$ *is a symmetric monoidal category.*

Proof. Since we have the statement analogous to Lemma 4.6, the condition b) of Definition 4.5 is again satisfied.

c) holds because of the nested relations associated to (A.1.8).

For a) we again use the identity relations to write the morphisms f, g such that parallel to , , , , , ,  which possibly contain further cobordisms there are only nested .

Now we distinguish two cases: If a basic cobordism adjoins to $R_{A,B}$, then we use one of the twist relations (A.1.8) - (A.1.11) or (A.3.11) to move it past $R_{A,B}$. If a nested basic cobordism adjoins to $R_{A,B}$, then we firstly use nested identity relations to move everything in the outer cobordism past $R_{A,B}$. After that we use a nested twist relation to move the outer cobordism with identities in it past $R_{A,B}$. Then we use again nested identity relations to put the cobordisms that were nested in the outer cobordism when we started back in. $\square$

Part II

Spaltenstein varieties

1 Spaltenstein varieties and first properties

In this section we state the fundamental definitions and discuss first properties.

Let V be an n-dimensional complex vector space and let $N: V \to V$ be a nilpotent endomorphism with two Jordan blocks.

Definition 1.1. A *partial flag of type* $(i_1, \dots, i_m)$ (where $0 < i_1 < \dots < i_m = n$) consists of subspaces F_{i_l} of V with $\dim F_{i_l} = i_l$ and $F_{i_1} \subset F_{i_2} \subset \dots \subset F_{i_m}$. The partial flags of type $(i_1, \dots, i_m)$ form a complex algebraic variety (see [Har92]) which we call $Fl(i_1, \dots, i_m)$.

A partial flag is called N-*invariant*, if $NF_{i_l} \subset F_{i_{l-1}}$ holds for all $l = 1, \dots, m$, where $F_{i_0} := \{0\}$.

The variety of N-invariant partial flags of type $(i_1, \dots, i_m)$ is called *Spaltenstein variety of type* $(i_1, \dots, i_m)$ and denoted $\mathrm{Sp}(i_1, \dots, i_m)$.

Lemma 1.2. *For an N-invariant flag of type $(i_1, \dots, i_m)$ we have*

$$\dim F_{i_l} \leq \dim F_{i_{l-1}} + 2,$$

hence $i_l - i_{l-1} \leq 2$.

Proof. Let $F_j \subset F_i$ such that $NF_j \subset F_i$ and suppose $\dim F_i > \dim F_j + 2$ holds. Then we have

$$\dim NF_i = \dim F_i - \dim \ker N|_{F_i} \geq \dim F_i - 2 > \dim F_j,$$

since N is of Jordan type $(n-k, k)$. Therefore, NF_i cannot be a subset of F_j, which is a contradiction. Thus, the assertion follows with $i = i_l$, $j = i_{l-1}$. □

Remark 1.3. Let $F_\bullet = (F_{j_1} \subset \dots \subset F_{j_r})$ be an N-invariant flag. Let $\{j_1, \dots, j_r = n\}$ be a subset of $\{i_1, \dots, i_m\}$. Then $F'_\bullet = (F'_{i_1} \subset \dots \subset F'_{i_m})$ with $F'_s = F_s$ if $s = j_l = i_{l'}$ and F'_s arbitrary otherwise is an N-invariant flag as well.

Indeed, consider NF_{i_a}. If $i_a \in \{j_1,\dots,j_r\}$ holds, i.e. $i_a = j_l$ for some l, we have $NF'_{i_a} = NF_{j_l} \subset F_{j_{l-1}} = F'_{i_b}$ for some $b < a$. Now, in particular we have $NF'_{i_a} \subset F'_{i_b} \subset F'_{i_{a-1}}$.
If we have $i_a \notin \{j_1,\dots,j_r\}$, then there are (since $j_r = n$) $i_d, i_b \in \{j_1,\dots,j_r\}$ with $d > a > b$ and $NF'_{i_d} = NF_{j_l} \subset F_{j_{l-1}} = F'_{i_b}$, where $j_{l-1} = i_b = 0$ is possible. Hence, $NF'_{i_a} \subset NF'_{i_d} \subset F'_{i_b}$. Now, we get again $NF'_{i_a} \subset F'_{i_b} \subset F'_{i_{a-1}}$.

Definition 1.4. Let $(i_1,\dots,i_m) \in \mathbb{N}^m$ with $0 < i_1 < \cdots < i_m = n$. A *tableau of shape* $(n-k,k)$ *of type* $(i_1,\dots,i_m)$ is a Young diagram of shape $(n-k,k)$ filled with $(i_l - i_{l-1})$-times the entry i_l for $l = 1,\dots,m$, where $i_0 := 0$.

In the following all tableaux will be of shape $(n-k,k)$ unless stated otherwise.

A *row strict tableau of type* $(i_1,\dots,i_m)$ is a tableau of type $(i_1,\dots,i_m)$ with strictly decreasing entries in the rows.

A *standard tableau of type* $(i_1,\dots,i_m)$ is a row strict tableau of type $(i_1,\dots,i_m)$ with decreasing entries in the columns.

Remark 1.5. This is similar to the usual definition of semi-standard tableaux. However, we work with (strictly) decreasing rows respectively columns instead of (strictly) increasing.

Example 1.6. Here is an example for a row strict tableau w and a standard tableau S, respectively, of type $(1,3,4,5)$ and shape $(n-k,k)$ for $n = 5, k = 2$:

$$w = \begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 5 & 3 \\ \cline{1-2}\end{array} \qquad S = \begin{array}{|c|c|c|}\hline 5 & 3 & 1 \\ \hline 4 & 3 \\ \cline{1-2}\end{array}$$

Remark 1.7. Note that because of the strictly decreasing rows every number appears at most twice in a row strict tableau and thus also in a standard tableau since we only have two rows. Consequently, we get $i_{k+1} - i_k \leq 2$. Note that this is the property that was proven in Lemma 1.2 for indexing set of the Spaltenstein variety.

2 Reduction to Springer fibres

In this section we explain the theorems of Spaltenstein and Fung which we want to generalise. The basis for this generalisation is laid by proving a 1-1 correspondence between irreducible components of Spaltenstein varieties and a certain subset of standard tableaux of type $(1,\dots,n)$.

Definition 2.1. A Spaltenstein variety of type $(1,\dots,n)$ is called *Springer fibre.*

In the following, for N of Jordan type $(n-k,k)$ let $Y_{n,k}$ be the Springer fibre and let $\mathcal{S}_{n,k}$ the set of all standard tableaux of type $(1,\dots,n)$ and shape $(n-k,k)$. Let $Y_n = \coprod_k Y_{n,k}$ and $\mathcal{S}_n = \bigcup_k \mathcal{S}_{n,k}$. Now we inductively construct a map

$$\begin{array}{rcl} \pi : Y_n & \to & \mathcal{S}_n \\ (F_\bullet, N) & \mapsto & \sigma_{F,N} \end{array} \tag{2.1}$$

as follows:
For $n = 1$ we have $N = (0)$ and $F_\bullet = \mathbb{C}^1$. We define $\sigma_{F_\bullet,N} = \boxed{1} \in \mathcal{S}_{1,0}$.

Let $(F_\bullet, N) \in Y_n$, $F_\bullet = (F_1 \subset \cdots \subset F_n)$. Then $F_\bullet/F_1 = (F_2/F_1 \subset \cdots \subset F_n/F_1)$ is N'-invariant for $N' : \mathbb{C}^n/F_1 \to \mathbb{C}^n/F_1$ being the map induced by N. Therefore, we have $(F_\bullet/F_1, N') \in Y_{n-1}$. Hence, $\sigma_{F_\bullet/F_1,N'} \in \mathcal{S}_{n-1}$ is already constructed. Now we construct $\sigma_{F,N}$ by adding 1 to all numbers in $\sigma_{F/F_1,N'}$ and putting a 1 in the additional box of N. By construction, $\sigma_{F,N}$ is a standard tableaux since the succession of the numbers is not changed by adding 1.

For example, we have $\begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 2 & & \\ \hline \end{array} \rightsquigarrow \begin{array}{|c|c|c|}\hline 5 & 4 & 2 \\ \hline 3 & 1 \\ \cline{1-2} \end{array}$.

One can see that π even determines a map $\pi|_{Y_{n,k}} : Y_{n,k} \to \mathcal{S}_{n,k}$, which is the map Spaltenstein constructed in [Spa76].

Remark 2.2. The Young diagram of N' is constructed from the Young diagram of N by deleting one of the boxes that are rightmost in the corresponding row and at the bottom of the corresponding column.

From now on we fix $n \geq 2k$ and write Y and $\mathcal{S}$ instead of $Y_{n,k}$ and $\mathcal{S}_{n,k}$. Furthermore, we fix a nilpotent matrix N of Jordan type $(n-k, k)$. Explicitly, we equip V with an ordered basis $\{e_1, \dots e_{n-k}, f_1, \dots f_k\}$ with the action of N defined by $N(e_i) = e_{i-1}, N(f_i) = f_{i-1}$, where by convention, $e_0 = f_0 = 0$.

Spaltenstein showed:

Theorem 2.3. ([Spa76])
Let π be defined as in formula (2.1). Then the set of standard tableaux of type $(1, \dots, n)$ is in bijective correspondence with the irreducible components of the Springer fibre Y.
This bijection is given by $\sigma \mapsto \overline{\pi|_Y^{-1}(\sigma)} =: Y_\sigma$.

This theorem of Spaltenstein holds even if N is not an endomorphism of Jordan type $(n-k, k)$ and the standard tableaux are not of shape $(n-k, k)$, but more generally for any nilpotent N. In general, it is complicated to calculate the closure $\overline{\pi|_Y^{-1}(\sigma)}$. But in our special case, where we only have two Jordan blocks, Fung explicitly determined how the irreducible components associated with a standard tableau look like:

Theorem 2.4. ([Fun03, Theorem 5.2])
Let N be a nilpotent map of Jordan type $(n-k, k)$, and let σ be a standard tableau of shape $(n-k, k)$. Then the component Y_σ of the Springer fibre Y consists of all flags whose subspaces satisfy the following conditions:

- *for each i*
$$F_i \subset N^{-1}(F_{i-1})$$
- *if i is on the top row of the tableau σ and $i-1$ is on the bottom row, then*
$$F_i = N^{-1}(F_{i-2})$$

- *if i and $i-1$ are both in the top row of σ, then*
  - *if $F_{i-1} = N^{-d}(F_r)$ where r is on the bottom row, then*
    $$F_i = N^{-d-1}(F_{r-1})$$
  - *if $F_{i-1} = N^{-d}(\operatorname{im} N^{n-k-j})$ where $0 \le j < n-2k$, then*
    $$F_i = N^{-d}(\operatorname{im} N^{n-k-j-1}).$$

(Here 0 is thought of being in the top row, $\{0\} = F_0 = \operatorname{im} N^{n-k}$)

Remark 2.5. There is a typing error in Fung's paper: If we have $k < a \le n-k$ and then replace a by $n-k-j$, we get for j the inequality $n-2k > j \ge 0$ and not $k > j \ge 0$.

We want to generalise Fung's theorem to Spaltenstein varieties. For this purpose, we first generalise Spaltenstein's theorem.

Definition 2.6. Let $\widetilde{Y}$ be a Spaltenstein variety of type $(i_1, \dots, i_m)$. Let $F_\bullet = (F_{i_1} \subset \dots \subset F_{i_m}) \in \widetilde{Y}$. We call the set

$$X = X(F) := \{F'_\bullet = (F'_1 \subset F'_2 \subset \dots \subset F'_n) : F'_{i_l} = F_{i_l} \forall\, l = 1, \dots, m\}$$

the set of the *full flags associated with the partial flag F*.

Definition 2.7. ([Spa76, p. 455])
Let $I \subset \{1, \dots, n-1\}$. Then a *subspace of type I* is a set of flags in the flag variety of the form

$$\{(F_1 \subset \dots \subset F_n) : F_j \text{ is fixed for all } j \in \{1, \dots, n\} \setminus I\}.$$

We call Z an *I-variety* if it is a union of subspaces of type I.

Remark 2.8. For $I \subset J$ every J-variety is an I-variety as well. Indeed, because of $\{1, \dots, n\} \setminus J \subset \{1, \dots, n\} \setminus I$ we have that every subset of type J can be written as a union of subsets of type I.

Example 2.9. a) Let $F_\bullet$ be a partial flag in the Spaltenstein variety $\widetilde{Y}$ of type $(i_1, \dots, i_m)$. Then the set X of full flags associated with the partial flag $F_\bullet$ is a subspace of type $I = \{1, \dots, n-1\} \setminus \{i_1, \dots, i_m\}$ in the Springer fibre Y.

Indeed, we have $X \subset Y$ because of Remark 1.3 and the rest is immediately clear from the definitions.

b) For σ a standard tableau let $I_\sigma = \{i | \sigma_i \le \sigma_{i+1}\}$ where σ_i is the number of the column of σ in which we find i. Spaltenstein showed in [Spa76, S. 455] that Y_σ is an I_σ-variety and I_σ is maximal with respect to inclusion with this property.

Lemma 2.10. *Let Y be the Springer fibre.*

a) Let U be a subspace of type $I = \{1, \dots, n-1\} \setminus \{j_1, \dots, j_r\}$ in the Springer fibre Y. Then we have $NF_{j_l} \subset F_{j_{l-1}}$ for every $F_\bullet = (F_1, \dots, F_n) \in U$.

b) Let U be a subspace of type I in Y. Then there are no consecutive numbers in I.

Proof. Let $a > b \in \{1,\dots n\} \smallsetminus I$ and assume all intermediate numbers are in I. Then for all $F_\bullet = (F_1 \subset \cdots \subset F_n) \in U$ we have that F_a and F_b are fixed and all possibilities for $F_{a-1},\dots,F_{b+1}$ with $F_a \supset F_{a-1} \supset \cdots \supset F_{b+1} \supset F_b$ appear. Since $F \in U \subset Y$, we have $NF_a \subset F_{a-1}$ for all possible choices of F_{a-1}, thus NF_a lies in the intersection of all F_{a-1}. Since the subsets between F_b and F_{a-1} including F_{a-1} are all not fixed, we have

$$F_b = \bigcap_{F_{a-1}: F_a \supset F_{a-1} \supset \cdots \supset F_b} F_{a-1} \supset NF_a.$$

Consequently, we get a). On the other hand, we conclude

$$\dim F_b \geq \dim NF_a \geq \dim F_a - 2, \tag{2.2}$$

where the last inequality holds, as in Lemma 1.2, because N is of Jordan type $(n-k,k)$.

If we have $a > b+2$, we get $\dim F_a > \dim F_b + 2$, which contradicts (2.2). Thus, b) holds. □

Lemma 2.11. ([Spa76, p. 455])
Let Y be the Springer fibre. Any subspace of type I contained in Y is contained in an irreducible component which is an I-variety.

We use the results of Spaltenstein above to show the following theorem, which is stated in a different notation in [Spa76] without a proof.

Theorem 2.12. *Let $\widetilde{Y}$ be a Spaltenstein variety of type $(i_1,\dots,i_m)$. Let $I := \{1,\dots,n-1\} \smallsetminus \{i_1,\dots,i_m\}$, let $\mathcal{S}$ be the set of standard tableaux of type $(1,2,\dots,n)$ and $I_\sigma = \{i | \sigma_i \leq \sigma_{i+1}\}$ for $\sigma \in \mathcal{S}$, where σ_i is the column number of σ containing i. Then there is a canonical bijection*

$$\{\textit{irreducible components of } \widetilde{Y}\} \overset{1:1}{\longleftrightarrow} \mathcal{S}_I := \{\sigma \in \mathcal{S} | I \subset I_\sigma\}.$$

Proof.

- The map

$$\begin{array}{rccc} \mathrm{pr}: & Z := \bigcup_{I \subset I_\sigma} Y_\sigma & \to & \widetilde{Y} \\ & (F_1 \subset \cdots \subset F_n) & \mapsto & (F_{i_1} \subset \cdots \subset F_{i_m}) \end{array}$$

  given by forgetting the subsets of the full flag with indices in I is well-defined: Let $F_\bullet = (F_1 \subset \cdots \subset F_n) \in Z$, so $F_\bullet \in Y_\sigma$ for a σ with $I \subset I_\sigma$. By Example 2.9 b) Y_σ is an I_σ-variety, thus $F_\bullet$ is contained in a subspace of type I_σ. By Lemma 2.10 a) we have $NF_{j_l} \subset F_{j_{l-1}}$ for $l = 1,\dots,r$ and $I_\sigma = \{1,\dots,n-1\} \smallsetminus \{j_1,\dots,j_r\}$. Because of $I \subset I_\sigma$ we have $\{j_1,\dots,j_r\} \subset \{i_1,\dots,i_m\}$, and by Remark 1.3, $(F_{i_1} \subset \cdots \subset F_{i_m}) = \mathrm{pr}(F)$ is an N-invariant flag.

- pr is surjective:
  Let $F_\bullet \in \widetilde{Y}$ and let X be the set of associated full flags. By Example 2.9 a) X is a subspace of type I in Y. Hence, by Lemma 2.11 there exists an irreducible component Y_σ with $X \subset Y_\sigma$, where Y_σ is an I-variety. Therefore, we have $I \subset I_\sigma$ because of the maximality of I_σ. So we have

$$F \in \mathrm{pr}(X) \subset \mathrm{pr}(Y_\sigma) \subset \mathrm{pr}\left(\bigcup_{I \subset I_\sigma} Y_\sigma\right).$$

- pr is a morphism of varieties:
  By [Spa82, S. 82] $\widehat{\mathrm{pr}} : Fl(1,\dots,n) \to Fl(i_1,\dots,i_m)$, given by forgetting the subsets F_j with $j \notin \{i_1,\dots,i_m\}$, is a morphism of varieties. By [Hum75, 21.1 e)] $Fl(1,\dots,n)$ is complete. The Springer fibre Y is a closed subvariety of $Fl(1,\dots,n)$, and the components Y_σ are closed in Y. Thus, $\bigcup_{I \subset I_\sigma} Y_\sigma$ is a closed subvariety of $Fl(1,\dots,n)$, and so by [Hum75, 21.1 a)] it is complete. Therefore, $\mathrm{pr} : Z \to Fl(i_1,\dots,i_m)$ is as restriction of $\widehat{\mathrm{pr}}$ a morphism of varieties. By [Hum75, 21.1 b)] $\mathrm{pr}(Z)$ is closed, and because of the surjectivity we have $\mathrm{pr}(Z) = \widetilde{Y}$. Hence, $\mathrm{pr} : Z \to \widetilde{Y}$ is a morphism of varieties.

- $\mathrm{pr}(Y_\sigma)$ is irreducible:
  Since pr is continuous and Y_σ irreducible, the assertion follows from [Hum75, Proposition 1.3 A b)].

- $\mathrm{pr}(Y_\sigma)$ is closed:
  Each Y_σ is an irreducible component and hence closed in Z. Consequently, it is complete by [Hum75, 21.1 a)]. Thus, the assertion follows from [Hum75, 21.1 b)].

- For $I \subset I_\sigma$ we have that Y_σ is an irreducible component of Z:
  Each Y_σ is irreducible and closed and not contained in any other $Y_{\sigma'}$. In addition, there are only finitely many Y_σ's. Since Z is the union of the Y_σ and these conditions are equivalent to being maximal irreducible, the Y_σ are irreducible components of Z.

- The irreducible components of Z are mapped to irreducible components of $\widetilde{Y}$:
  As shown above, $\mathrm{pr}(Y_\sigma)$ is irreducible and closed. The map pr is surjective, so

$$\widetilde{Y} = \mathrm{pr}(Z) = \mathrm{pr}\left(\bigcup_{I \subset I_\sigma} Y_\sigma\right) = \bigcup_{I \subset I_\sigma} \mathrm{pr}(Y_\sigma).$$

  Therefore, $\mathrm{pr}(Y_\sigma)$ is an irreducible component. Since Y_σ is an I-variety, we have $\mathrm{pr}^{-1}(\mathrm{pr}(Y_\sigma)) \subset Y_\sigma$, and thus $\mathrm{pr}^{-1}(\mathrm{pr}(Y_\sigma)) = Y_\sigma$, because the other inclusion always holds. Therefore, the $\mathrm{pr}(Y_\sigma)$ are distinct.

Altogether, there is a 1-1 correspondence between the irreducible components of $\widetilde{Y}$ and the set $\{Y_\sigma | I \subset I_\sigma\}$, thus also between the irreducible components of $\widetilde{Y}$ and the set $\{\sigma \in \mathcal{S} | I \subset I_\sigma\} = \mathcal{S}_I$. □

The bijection from Theorem 2.12 will be made explicit in the next section.

3 Description of irreducible components

In this section we use the theorem from last chapter to prove a 1-1 correspondence between irreducible components of Spaltenstein varieties of type $(i_1,\ldots,i_m)$ and standard tableaux of type $(i_1,\ldots,i_m)$. Then we prove a generalisation of Fung's theorem which describes the structure of the irreducible components.

Definition 3.1. Let S be a standard tableau of type $(i_1,\ldots,i_m)$. We associate a standard tableau S' of type $(1,\ldots,n)$ with S as follows: By definition, in a standard tableau of type $(i_1,\ldots i_m)$ there are at most two entries i_j for all j. If there are two entries i_j, then they have to be in different rows. Hence, we can associate a unique standard tableau of type $(1,\ldots,n)$ with a standard tableau of type $(i_1,\ldots,i_m)$ by changing the entry i_j in the lower row to i_j-1 if there are two such entries in the tableau. This is possible because if i_j is a double entry, then there is no i_j-1 in the tableau.

In this way we get an injective map

$$\begin{aligned}\varphi:\quad &\{\text{standard tableaux of type } (i_1,\ldots,i_m)\}\\ &\quad\hookrightarrow \{\text{standard tableaux of type } (1,\ldots,n)\}.\end{aligned}$$

Example 3.2. For $n=8$ and $k=3$ the standard tableau $\begin{array}{|c|c|c|c|c|}\hline 8&6&4&3&1\\\hline 7&6&3\\\cline{1-3}\end{array}$ of type $(1,3,4,6,7,8)$ is mapped to $\begin{array}{|c|c|c|c|c|}\hline 8&6&4&3&1\\\hline 7&5&2\\\cline{1-3}\end{array}$ by φ.

Theorem 3.3. *The irreducible components of the Spaltenstein variety of type $(i_1,\ldots,i_m)$ are in bijective correspondence with the standard tableaux of type $(i_1,\ldots,i_m)$ and shape $(n-k,k)$.*

Proof. Consider the set $I_\sigma = \{i|\sigma_i \le \sigma_{i+1}\}$ for $\sigma\in\mathcal{S}$. Again, σ_i is the number of the column in which i lies in σ. Consider $i\in I_\sigma$. In the following we find an equivalent description for I_σ. If i and $i+1$ occur in the same row, then $\sigma_{i+1}<\sigma_i$ by row strictness. Hence, $i\in I_\sigma$ implies that i and $i+1$ are in different rows.

Assume, i is in the top row. If there is no number below i and $i+1$ is in the bottom row, then we deduce $\sigma_i>\sigma_{i+1}$ and $i+1$ cannot be in the bottom row. So let a be the number that is in the bottom row in the same column as i in σ. By the column ordering in the standard tableau, we have $a\le i$. By row strictness we have $\sigma_b<\sigma_a=\sigma_i$ for all $b>a$ that are in the bottom row. So, because of $\sigma_i\le\sigma_{i+1}$, we conclude that $i+1$ cannot be in the bottom row. In the first paragraph of the proof, we showed that i and $i+1$ do not occur in the same row, so i has to be in the bottom row and $i+1$ in the top row.

Conversely, if i is in the bottom row and $i+1$ in the top row, then $\sigma_i>\sigma_{i+1}$ is not possible. Indeed, in this case there has to be an a in the same column as $i+1$ in the bottom row with $i+1\ge a>i$. This is a contradiction.

Hence,

$$I_\sigma = \{i|i \text{ is in the bottom row of } \sigma \text{ and } i+1 \text{ in the top row}\}$$

and

$$\mathcal{S}_I = \{\sigma \in \mathcal{S} | i \text{ occurs in the bottom and } i+1 \text{ in the top row of } \sigma \text{ resp. } \forall i \in I\}.$$

Now consider the standard tableaux of type $(i_1, \dots, i_m)$. The map φ from Definition 3.1 is injective and its image consists of the standard tableaux of type $(1, \dots, n)$ with i_j in the top row and $i_j - 1$ in the bottom row for all $j = 1, \dots, m$ with i_j a double entry. So these standard tableaux of type $(1, \dots, n)$ are in 1-1 correspondence to the standard tableaux of type $(i_1, \dots, i_m)$.
Since $I = \{1, \dots, n-1\} \setminus \{i_1, \dots, i_m\}$, we have $i \in I$ if and only if $i = i_j - 1$ for i_j a double entry. Consequently, the set of standard tableaux of type $(i_1, \dots, i_m)$ is in bijection to $\mathcal{S}_I$, therefore by Theorem 2.12 it is in bijection to the set of irreducible components of $\widetilde{Y}$. □

Definition 3.4. We denote by $\widetilde{Y}_S$ the irreducible component of the Spaltenstein variety $\widetilde{Y}$ corresponding to a standard tableau S of type $(i_1, \dots, i_m)$. This exists and is unique by Theorem 3.3.

To summarise, we obtain this bijection as follows: Every irreducible component of $\widetilde{Y}$ is the image of an irreducible component Y_σ via pr. To Y_σ we assign a standard tableau σ of type $(1, \dots, n)$ via the Spaltenstein-Vargas bijection from Theorem 2.3. As shown in the last proof, σ is in the image of φ and thus corresponds to a standard tableau S of type $(i_1, \dots, i_m)$.

Remark 3.5. From the proofs of Theorem 2.12 and Theorem 3.3 we particularly get the following: If $F_\bullet \in \widetilde{Y}_S$, then all full flags associated with $F_\bullet$ are in $Y_{\varphi(S)}$, i.e. $\mathrm{pr}^{-1}(F_\bullet) \subset Y_{\varphi(S)}$. On the other hand, for a full flag $F'_\bullet$ with $F'_\bullet \in Y_\sigma$ such that $I \subset I_\sigma$ we know that the projected partial flag lies in $\widetilde{Y}_{\varphi^{-1}(\sigma)}$.

Remark 3.6. For i_l is a double entry in S we have $F_{i_l} = N^{-1} F_{i_{l-1}}$. Indeed, because of the N-invariance we have $F_{i_l} \subset N^{-1} F_{i_{l-1}}$. Furthermore, for W a subset of V consider the map $N|_{N^{-1}W} : N^{-1}W \to N^{-1}W$. We deduce

$$\dim N^{-1}W = \dim \operatorname{im} N|_{N^{-1}W} + \dim \ker N|_{N^{-1}W} \leq \dim W + 2.$$

Since $i_{l-1} = i_l - 2$, for dimensional reasons we get the assertion.

Theorem 3.7 (Explicit description of irreducible components)**.**
Let S be a standard tableau of type $(i_1, \dots, i_m)$. Then the irreducible component $\widetilde{Y}_S$ of the Spaltenstein variety $\widetilde{Y}$ of type $(i_1, \dots, i_m)$ consists of all flags, whose subspaces satisfy the following conditions:

- *for each l*
$$F_{i_l} \subset N^{-1}(F_{i_{l-1}})$$
- *if i_l is on the top row of the tableau S and $i_l - 1$ is in the bottom row and $i_l - 1$ is not a double entry, then*
$$F_{i_l} = N^{-1}(F_{i_l - 2})$$

- *if i_l and $i_l - 1$ are both in the top row of S, then*
  - *if $F_{i_l-1} = N^{-d}(F_r)$ where r is in the bottom row and not a double entry, then*
  $$F_{i_l} = N^{-d-1}(F_{r-1})$$
  - *if $F_{i_l-1} = N^{-d}(\operatorname{im} N^{n-k-j})$ where $0 \leq j < n-2k$, then*
  $$F_{i_l} = N^{-d}(\operatorname{im} N^{n-k-j-1}).$$

(Here 0 is thought of being in the top row, $\{0\} = F_0 = \operatorname{im} N^{n-k}$)

For the proof of the theorem and in the following we need the notion of dependent and independent subsets.

Definition 3.8. If for all flags satisfying the conditions the subset F_{i_l} is specified as $F_{i_l} = N^{-j}(F_{i_s})$ for some $j > 0$ or $F_{i_l} = N^{-j}(\operatorname{im} N^t)$ for some $j \geq 0$, it is called *dependent.* If a subset is not dependent, it is called *independent.*

Proof of Theorem 3.7. Let $F_\bullet = (F_{i_1} \subset \cdots \subset F_{i_m}) \in \widetilde{Y}_S$. We want to show that the above conditions are met. Let $\widehat{F}_\bullet \in X(F_\bullet)$ be an associated full flag. By Remark 3.5 we have $\widehat{F}_\bullet \in Y_{\varphi(S)}$. Therefore, $\widehat{F}_\bullet$ meets the conditions of [Fun03, Theorem 5.2] for $i = 1, \dots, n$, so in particular for $i_1, \dots, i_m$. Here, the first condition holds for $i_{l-1} = i_l - 1$ on the nose and for $i_{l-1} = i_l - 2$ it holds because of the second condition of [Fun03, Theorem 5.2] we even have equality.

By Remark 3.6 $F_{i_l} = N^{-1}(F_{i_l-2})$ holds for all i_l which are a double entry in S. Thus, the second condition above is the one remaining necessary after removing the redundant part for double entries. In addition, we get that r is no double entry because otherwise it is in the top row of $\varphi(S)$. Furthermore, for i_l a double entry no F_{i_r} is dependent on F_{i_l-1}, because i_l is in the top of $\varphi(S)$ and $i_l - 1$ in the bottom row. Hence, the above conditions remain.

Let $F_\bullet \in \widetilde{Y}$ be a partial flag satisfying the conditions above. Let $\widehat{F}_\bullet$ be a full flag associated with $F_\bullet$ and let $\widehat{S} = \varphi(S)$. Now we have $F_i \subset N^{-1}F_{i-1}$ for all $i \in \widehat{S}$. All other conditions of [Fun03, Theorem 5.2] are also fulfilled: If $i = i_l$ and $i_l - 1$ is not a double entry in S, then the conditions coincide. If $i = i_l$ and $i_l - 1$ is a double entry, we use the condition for $i_l - 1$ in the top row and the conditions concur again. If $i = i_l$ is a double entry, then $i - 1$ is in the bottom row of $\widehat{S}$ and since $F_i = N^{-1}(F_{i-1})$ by Remark 3.6 the conditions coincide. If $i \notin S$, then i cannot be in the top row of $\widehat{S}$ by definition of φ.

Thus, we have $\widehat{F}_\bullet \in Y_{\varphi(S)}$, and by Remark 3.5 we have $F_\bullet = \operatorname{pr}(\widehat{F}_\bullet) \in \widetilde{Y}_S$. □

4 Fixed points

In this section we study fixed points of an action of $\mathbb{C}^*$ on $Sp(i_1, \dots, i_m)$ and use them to define generalised irreducible components. We then use cup diagrams to describe a correspondence between generalised irreducible components and irreducible components. At the end we consider a map from a generalised irreducible component of a Spaltenstein variety to a generalised irreducible component of a certain Springer fibre.

Remark 4.1 (Origin of the $\mathbb{C}^*$-action)**.**
Let $T \cong (\mathbb{C}^*)^n$ be the torus of diagonal matrices in the basis given by the e_i's and f_i's. T acts on the partial flag variety $Fl(i_1,\dots,i_m)$ via its action on the e_i's and f_i's.

For $t \in T$ acting on the Spaltenstein variety as well, it has to commute with N. For $t = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ we have $Nt = tN$ if and only if $\lambda_1 = \dots = \lambda_{n-k}$ and $\lambda_{n-k+1} = \dots = \lambda_n$.
Therefore, the part of T commuting with N is isomorphic to $(\mathbb{C}^*)^2$.

Now we choose the cocharacter

$$\begin{aligned} \mathbb{C}^* &\to (\mathbb{C}^*)^2 \\ t &\mapsto (t^{-1}, t) \end{aligned}$$

and get an action of $\mathbb{C}^*$ on $\mathrm{Sp}(i_1,\dots,i_m)$.

Lemma 4.2. *There is a $\mathbb{C}^*$-action on $\mathrm{Sp}(i_1,\dots,i_m)$ such that*

$$\{\text{fixed points of the action}\} \overset{1:1}{\underset{\Psi}{\longleftrightarrow}} \{\text{row strict tableaux of type } (i_1,\dots,i_m)\}.$$

Proof. We define the $\mathbb{C}^*$-action by $t.e_i = t^{-1}e_i, t.f_i = tf_i$ (as in the above Remark). Now, we define Ψ by $w \mapsto \mathcal{F}_\bullet(w)$, where w is a row strict tableau and $\mathcal{F}_\bullet(w)$ the partial flag with $F_{i_l}(w) = \langle\{e_j, f_r | j \le t_{i_l}, r \le b_{i_l}\}\rangle$, where t_s is the number of indices smaller than or equal to s in the top row and similarly for b_s in the bottom row.

$\mathcal{F}_\bullet(w)$ is a fixed point under the action. Indeed, for $x \in F_{i_l}(w)$ we have $x = \sum_{j=1}^{t_{i_l}} a_j e_j + \sum_{r=1}^{b_{i_l}} c_r f_r$. Thus, $t.x = \sum_{j=1}^{t_{i_l}} a_j t^{-1} e_j + \sum_{r=1}^{b_{i_l}} c_r t f_r = \sum_{j=1}^{t_{i_l}} (a_j t^{-1}) e_j + \sum_{r=1}^{b_{i_l}} (c_r t) f_r \in F_{i_l}$.
By definition, $F_{i_l}(w) = \langle\{e_j, f_r | j \le t_{i_l}, r \le b_{i_l}\}\rangle$ and $Ne_j = e_{j-1}$, $Nf_r = f_{r-1}$, so we have

$$\begin{aligned} NF_{i_l}(w) &= \langle\{e_j, f_r | j \le t_{i_l} - 1, r \le b_{i_l} - 1\}\rangle \\ &\subset \langle\{e_j, f_r | j \le t_{i_{l-1}}, r \le b_{i_{l-1}}\}\rangle = F_{i_{l-1}}(w). \end{aligned}$$

The inclusion holds, since by Lemma 1.2 $i_{l-1} = i_l - 1$ or $i_{l-1} = i_l - 2$ and in the second case there is a i_l in both lines because of row-strictness. Therefore, we have $\mathcal{F}_\bullet(w) \in \widetilde{Y}$.

Ψ is injective, since $\mathcal{F}_\bullet(w) = \mathcal{F}_\bullet(w')$ implies $F_{i_l}(w) = F_{i_l}(w')$ and hence $t_{i_l} = t'_{i_l}$ and $b_{i_l} = b'_{i_l}$. Inductively we get $w = w'$.

Ψ is surjective:
Let $F_\bullet = (F_{i_1} \subset \dots \subset F_{i_m})$ be a fixed point under the above action. We construct a row strict tableau w of type $(i_1,\dots,i_m)$ with $F_{i_l}(w) = F_{i_l}$ by induction on l.

$l = 1$: By Lemma 1.2 we have $\dim F_{i_1} \le 2$. Since $NF_{i_1} \subset F_0 = \{0\}$, we have $F_{i_1} \subset \ker N = \langle e_1, f_1\rangle$.

If $\dim F_{i_1} = 2$ we have $F_{i_1} = \langle e_1, f_1\rangle$. We put a 2 rightmost in both lines of the tableau w. Thus, we have

$$F_{i_1}(w) = F_2(w) = \langle e_1, f_1\rangle = F_{i_1}.$$

If $\dim F_{i_1} = 1$, since it is a fixed point and $F_{i_1} = \langle \lambda e_1 + \mu f_1 \rangle$ we get $F_{i_1} = \langle e_1 \rangle$ or $F_{i_1} = \langle f_1 \rangle$. We put a 1 rightmost in the tableau w, in the top row if $F_{i_1} = \langle e_1 \rangle$ and in the bottom row if $F_{i_1} = \langle f_1 \rangle$. In this case we also get $F_{i_1}(w) = F_{i_1}$.

Now assume the numbers up to i_l are already put into the tableau w such that $F_{i_l} = F_{i_l}(w) = \langle e_1, \dots, e_r, f_1, \dots, f_s \rangle$. For $F_{i_{l+1}}$ we have

$$F_{i_l} \subset F_{i_{l+1}} \subset N^{-1} F_{i_l} = \langle e_1, \dots, e_{r+1}, f_1, \dots, f_{s+1} \rangle .$$

If $\dim F_{i_{l+1}} = \dim F_{i_l} + 2$, we put a i_{l+1} in each row in the rightmost empty box of w. Again we have $F_{i_{l+1}}(w) = F_{i_{l+1}}$.

If $\dim F_{i_{l+1}} = \dim F_{i_l} + 1$, we have $F_{i_{l+1}} = F_{i_l} + \langle \lambda e_{r+1} + \mu f_{s+1} \rangle$ and because of the fixed point property we deduce again $\lambda = 0$ or $\mu = 0$. We put an i_{l+1} in the top row of w if $\mu = 0$ and we put it in the bottom row if $\lambda = 0$. So again we have $F_{i_{l+1}}(w) = F_{i_{l+1}}$.

The tableau w constructed this way is row strict, since we started in each row with a 1 or a 2 in the rightmost box and then gradually put bigger numbers left to the others. Altogether we have $F_\bullet = \mathcal{F}_\bullet(w)$ by construction. □

Definition 4.3. Let w be a row strict tableau of type $(i_1, \dots, i_m)$. Let $w_\vee$ be the set of numbers in the bottom row of the tableau, $w_\wedge$ the set of numbers in the top row and $w_\times$ the set of double entries.

We consider the sequence $\mathbf{a} = a_1 a_2 a_3 \dots a_n$, where $a_{i-1} = a_i = \times$ if $i \in w_\times$ and otherwise $a_i = \wedge$ if $i \in w_\wedge$ and $a_i = \vee$ if $i \in w_\vee$, and call it the *weight sequence* of w.

Associate with w a cup diagram $C(w)$ as follows: We consider the weight sequence and with the $\times$'s associate nothing. Then build the diagram inductively by adding an arc between any adjacent pair $\vee\wedge$, and then continuing the process for the sequence with these points (and the $\times$s) excluded. After that continue by matching all the remaining adjacent $\wedge\vee$-pairs and again ignoring all $\times$'s and the alreaddy connected points.

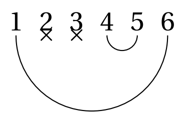
Now, several row strict tableaux of type $(i_1, \dots, i_m)$ have the same cup diagram. Among all the row strict tableaux which have the same cup diagram as w there is one standard tableau. This standard tableau can be constructed by putting every left endpoint of an arc in the cup diagram in the bottom row, every right endpoint or unmatched point in the top row and then inserting the double entries. Call this $S(w)$.

Example 4.4.

$$w = \begin{array}{|c|c|c|} \hline 5 & 3 & 1 \\ \hline 6 & 4 & 3 \\ \hline \end{array} \qquad\qquad S(w) = \begin{array}{|c|c|c|} \hline 6 & 5 & 3 \\ \hline 4 & 3 & 1 \\ \hline \end{array}$$

$$w_\wedge = \{1,3,5\}, \quad w_\vee = \{3,4,6\}, \quad w_\times = \{3\} = S(w)_\times,$$
$$S(w)_\wedge = \{3,5,6\}, \quad S(w)_\vee = \{1,3,4\}$$

weight sequence of w: $\wedge \times \times \vee \wedge \vee$, weight sequence of $S(w)$: $\vee \times \times \vee \wedge \wedge$

$C(w) =$  $= C(S(w))$

Definition 4.5. Let $P = \langle e_1, \dots, e_{n-k}\rangle$, $Q = \langle f_1, \dots, f_k\rangle$ and let $\widetilde{Y}$ be a Spaltenstein variety of type $(i_1, \dots, i_m)$. For each flag $F_\bullet$ in $\widetilde{Y}$, we can obtain a flag (with no longer necessarily distinct spaces) in P by taking the intersections $\mathcal{P}_i = F_i \cap P$, and similarly in Q by taking $\mathcal{Q}_i = \alpha(F_i/(F_i \cap P))$ with $\alpha : V/P \xrightarrow{\cong} Q$. We can define the new flag $F'_\bullet$ by putting $F'_i := \mathcal{P}_i + \mathcal{Q}_i \subset P \oplus Q = V$. Let $\widetilde{\mathcal{Y}}^0_w$ be the subvariety of partial flags $F_\bullet$ in $\widetilde{Y}$ with the property that $F'_\bullet = \mathcal{F}_\bullet(w)$ holds. Let $\widetilde{\mathcal{Y}}_w = \overline{\widetilde{\mathcal{Y}}^0_w}$ be its closure. If $(i_1, \dots, i_m) = (1, \dots, n)$ we write $\mathcal{Y}_w$ instead of $\widetilde{\mathcal{Y}}_w$. In the following, we call the $\widetilde{\mathcal{Y}}_w$ *generalised irreducible components*, even though this is not an official mathematical notation. But as one can see in the next theorem, the generalised irreducible components contain the irreducible components.

Theorem 4.6. *Let w be a row strict tableau of type $(i_1, \dots, i_m)$. Then $\widetilde{\mathcal{Y}}_w$ is the subset of $\widetilde{Y}_{S(w)}$ containing exactly the flags $F_\bullet$ which satisfy the following additional property: if $i \in (w_\wedge \cap S(w)_\vee) \setminus w_\times$, then $F_i = F_i(w)$.*

In particular, for any standard tableau S, we have $\widetilde{\mathcal{Y}}_S = \widetilde{Y}_S$.

Proof. First we confirm that these relations hold on $\widetilde{\mathcal{Y}}^0_w$ (and thus on $\widetilde{\mathcal{Y}}_w$, since they are closed conditions).

Consider first the case where $(i_1, \dots, i_m) = (1, \dots, i-1, i+1, \dots, n)$ for some i:
We use the map φ from Definition 3.1 for row strict tableaux as well. This is possible since in row strict tableaux the double entries also appear in different rows.

Let $F_\bullet \in \widetilde{\mathcal{Y}}^0_w$. Since $(F_1 \subset \dots \subset F_{i-1} \subset F_{i+1} \subset \dots \subset F_n)$ is N-invariant, $(F_1 \subset \dots \subset F_{i-1} \subset F_i \subset F_{i+1} \subset \dots \subset F_n)$ is also N-invariant for each possible F_i.

We show, that $(F_1 \subset \dots \subset F_n) \in \mathcal{Y}_{\varphi(w)}$:
By definition i is in the bottom row of $\varphi(w)$, so

$$F_i(\varphi(w)) = F_{i-1}(w) + \langle f_s\rangle,$$

where s is chosen such that $f_1, \dots, f_{s-1} \in F_{i-1}(w)$ and $F_{i+1}(w) = F_{i-1}(w) + \langle f_s, e_r\rangle$ for r chosen such that $e_1, \dots, e_{r-1} \in F_{i-1}(w)$.
Thus, since $F_\bullet \in \widetilde{\mathcal{Y}}^0_w$ we have

$$\mathcal{P}_{i-1} + \mathcal{Q}_{i-1} = (F_{i-1} \cap P) + \alpha(F_{i-1}/(F_{i-1} \cap P)) = F_{i-1}(w)$$

and

$$\mathcal{P}_{i+1} + \mathcal{Q}_{i+1} = (F_{i+1} \cap P) + \alpha(F_{i+1}/(F_{i+1} \cap P)) = F_{i+1}(w) = F_{i-1}(w) + \langle f_s, e_r\rangle.$$

Therefore, we have $f_s \in \mathcal{Q}_{i+1}$, and thus $f_s + (P \cap F_{i+1}) \in F_{i+1}/(F_{i+1} \cap P)$. Since $e_1, \dots, e_r \in \mathcal{P}_{i+1} = F_{i+1} \cap P$, we have

$$f'_s := f_s + \lambda_{r+1} e_{r+1} + \dots \lambda_{n-k} e_{n-k} \in F_{i+1}, \quad f'_s \notin F_{i-1}$$

for suitable $\lambda_{r+1}, \dots, \lambda_{n-k}$. We define

$$F_i := F_{i-1} + \langle f'_s\rangle.$$

Then we have $F_{i-1} \subset F_i \subset F_{i+1}$ and

$$\begin{aligned}\mathcal{P}_i + \mathcal{Q}_i &= F_i \cap P + \alpha(F_i/(F_i \cap P))\\ &= F_{i-1} \cap P + \alpha(F_{i-1}/(F_{i-1} \cap P) + \langle f_s + (P \cap F_{i-1})\rangle)\\ &= \mathcal{P}_{i-1} + \mathcal{Q}_{i-1} + \langle f_s \rangle = F_i(\varphi(w)).\end{aligned}$$

Therefore, $(F_1 \subset \cdots \subset F_n) \in \mathcal{Y}_{\varphi(w)}$.

Hence, by [SW08, Theorem 15] for these flags we have $F_j = F_j(\varphi(w))$ for all $j \in \varphi(w)_\wedge \cap S(\varphi(w))_\vee$. We have $F_j(\varphi(w)) = F_j(w)$ for all $j \neq i$, and $\varphi(w)_\wedge = w_\wedge, \varphi(S(w))_\vee = S(w)_\vee \cup \{i\} \setminus \{i+1\}$ holds by Definition of φ. Furthermore, we have $S(\varphi(w)) = \varphi(S(w))$, since φ maps $\times\times$ to $\vee\wedge$ in the associated cup diagrams and consequently, the associated standard tableaux have the same form.
In summary:

$$\varphi(w)_\wedge \cap S(\varphi(w))_\vee = w_\wedge \cap (S(w)_\vee \cup \{i\} \setminus \{i+1\}) = (w_\wedge \cap S(w)_\vee) \setminus \{i+1\}$$

since $i \notin w_\wedge$. Hence, the additional property follows.

Since $(F_1 \subset \cdots \subset F_n) \in \mathcal{Y}_{\varphi(w)}$, by [SW08, Theorem 15] we have $(F_1 \subset \cdots \subset F_n) \in Y_{S(\varphi(w))}$. Thus by Remark 3.5

$$(F_1 \subset \cdots \subset F_{i-1} \subset F_{i+1} \subset \cdots \subset F_n) \in \widetilde{Y}_{\varphi^{-1}(S(\varphi(w)))} = \widetilde{Y}_{S(w)}.$$

On the other hand, let $F_\bullet \in \widetilde{Y}_{S(w)}$ with $F_j = F_j(w)$ for $j \in (w_\wedge \cap S(w)_\vee) \setminus \{i+1\}$. Let $\widehat{F}_\bullet$ be a full flag associated with $F_\bullet$. Then by Remark 3.5 and $S(\varphi(w)) = \varphi(S(w))$ we get $\widehat{F}_\bullet \in Y_{S(\varphi(w))}$. As shown above we have

$$\varphi(w)_\wedge \cap S(\varphi(w))_\vee = (w_\wedge \cap S(w)_\vee) \setminus \{i+1\}.$$

Hence, $\widehat{F}_\bullet$ satisfies the conditions of [SW08, Theorem 15], and $\widehat{F}_\bullet \in \mathcal{Y}_{\varphi(w)}$ follows. In particular, we have $F_\bullet = \mathrm{pr}(\widehat{F}_\bullet) \in \widetilde{\mathcal{Y}}_w$.

For general $(i_1, \ldots, i_m)$ the reasoning is analogous, since the proof at most uses properties of F_{i-1} and F_{i+1}.

□

Definition 4.7. Let w be a row strict tableau of type $(i_1, \ldots, i_m)$. Let I be the set $\{1, \ldots, n\} \setminus \{i_1, \ldots, i_m\} =: \{j_1, \ldots, j_r\}$. We associate with w a row strict tableau of type $(1, \ldots, n-2r)$ as follows: We delete the boxes with double entries, i.e. those with $j_l + 1$, $l = 1, \ldots, r$. Then we replace the entries $a \in \{j_l + 2, \ldots, j_{l+1} - 1\}$ by $a - 2l$ for $l = 1, \ldots, r$. The result is still a row strict tableau, which contains the entries $1, \ldots, n-2r$ only once.

Thus, we get a map

$$\begin{aligned}p: &\{\text{row strict tableaux of type } (i_1, \ldots, i_m) \text{ of shape } (n-k, k)\}\\ &\to \{\text{row strict tableaux of type } (1, \ldots, n-2r) \text{ of shape } (n-k-r, k-r)\}\end{aligned}$$

For example

$$p\left(\begin{array}{|c|c|c|c|}\hline 6&5&4&3\\ \hline 7&3&1 \\ \cline{1-3}\end{array}\right)=\begin{array}{|c|c|c|}\hline 4&3&2\\ \hline 5&1 \\ \cline{1-2}\end{array}.$$

We define $\pi:\widetilde{\mathcal{Y}}_w\to\mathcal{Y}_{p(w)}$ by

$$\begin{aligned}&(F_1\subset\ldots\subset F_{j_1-1}\subset F_{j_1+1}\subset\cdots\subset F_{j_2-1}\subset\ldots\\ &\qquad\cdots\subset F_{j_l+1}\subset\cdots\subset F_{j_{l+1}-1}\subset\cdots\subset F_n)\\ \mapsto&(F_1\subset\ldots\subset F_{j_1-1}\subset NF_{j_1+2}\subset\cdots\subset NF_{j_2-1}\subset\ldots\\ &\qquad\cdots\subset N^lF_{j_l+2}\subset\cdots\subset N^lF_{j_{l+1}-1}\subset\cdots\subset N^rF_n)\end{aligned}$$

Theorem 4.8. *The map π from above is well-defined, bijective and a morphism of varieties.*

Proof. Since p as well as π are compositions of maps which only forget one index, it is enough to consider $(1,\ldots,i-1,i+1,\ldots,n)$ with $I=\{i\}$. In this case, the maps p and π are of the following kind:

$$\begin{aligned}p:&\{\text{row strict tableaux of type } (1,\ldots i-1,i+1,\ldots n)\}\\ &\to\{\text{row strict tableaux of type } (1,\ldots n-2) \text{ of shape } (n-k-1,k-1)\}\end{aligned}$$

is the map which sends a tableau to another one by deleting the boxes with $i+1$ in it and replacing the numbers $i+2,\ldots,n$ by $i,\ldots,n-2$.

π looks as follows

$$(F_1\subset\cdots\subset F_{i-1}\subset F_{i+1}\subset\cdots\subset F_n)\mapsto(F_1\subset\cdots\subset F_{i-1}\subset NF_{i+2}\subset\cdots\subset NF_n).$$

We first claim $(F_1\subset\cdots\subset F_{i-1}\subset NF_{i+2}\subset\cdots\subset NF_n)$ is a full flag:
As in Lemma 1.2 we have $\dim NF_j=\dim F_j-\dim\ker N|_{F_j}$ and $\dim\ker N|_{F_j}\leq 2$. By Remark 3.6 we have

$$F_{i+1}=N^{-1}F_{i-1}. \tag{4.1}$$

Thus, $\ker N|_{F_{i+1}}=2$ follows and also $\dim N|_{F_j}=2$ for $j\geq i+1$, since $F_{i+1}\subset F_j$ for $j\geq i+1$. Therefore, we get $\dim NF_j=\dim F_j-2$ for $j>i+1$. So, the dimensions fit. We have $F_{i-1}\subset NF_{i+2}$ because $NF_{i+1}=F_{i-1}$. For the remaining subset relation are clear or follow from $F_j\subset F_{j+1}$, which means $NF_j\subset NF_{j+1}$, for $j\geq i+1$.

Let $N'=N|_{NV}$. Then $(F_1\subset\cdots\subset F_{i-1}\subset NF_{i+2}\subset\cdots\subset NF_n)$ is N'-invariant for a N-invariant flag $(F_1\subset\cdots\subset F_{i-1}\subset F_{i+1}\subset\cdots\subset F_n)$, because:
For $j=1,\ldots,i-1$ we have $NF_j=N'F_j$ and the assertion follows right away. By assumption we have $NF_j\subset F_{j-1}$ for $j=i+3,\ldots,n$, hence $N'NF_j=NNF_j\subset NF_{j-1}$. Furthermore, $N'NF_{i+2}=NNF_{i+2}\subset NF_{i+1}=F_{i-1}$.

Let $(F_1\subset\cdots\subset F_{i-1}\subset F_{i+1}\subset\cdots\subset F_n)\in\widetilde{Y}_S$ for S a standard tableau. Then we have $(F_1\subset\cdots\subset F_{i-1}\subset NF_{i+2}\subset\cdots\subset NF_n)\in Y_{p(S)}$:

This follows from Theorem 3.7 and [Fun03, Theorem 5.2]. Indeed, for $j \leq i-1$ the conditions of Theorem 3.7 and Fung's Theorem are satisfied, since they do not change under p. We denote $\pi(F_\bullet)$ by $F'_\bullet$. Since $i+1$ is a double entry in w, if $i+2 \in w_\wedge$, then we have to distinguish the following cases:

If $i-1$ is in the bottom row in w (and hence in $p(w)$), then $F_{i+2} = N^{-2}F_{i-2}$ follows from Theorem 3.7 and (4.1). Therefore, we get

$$F'_i = NF_{i+2} = NN^{-2}F_{i-2} = N^{-1}F_{i-2} = N^{-1}F'_{i-2}$$

where the second last equality holds for dimensional reasons. So for i in the top row of $p(w)$ and $i-1$ in the bottom row, the conditions of Fung's theorem are satisfied.

If $i-1$ is not in the bottom row and $F_{i-1} = N^{-l}(F_r)$ for r in the bottom row, we know by construction that $i-1 = r+2l$. Furthermore by (4.1) we have $F_{i+1} = N^{-l-1}(F_r)$ for r in the bottom row, thus by Theorem 3.7 we have $F_{i+2} = N^{-l-2}(F_{r-1})$. Now we conclude

$$F'_i = NF_{i+2} = NN^{-l-2}(F_{r-1}) \subset N^{-l-1}(F_{r-1}) = N^{-l-1}(F'_{r-1}).$$

But since $2l+2+r-1 = i$ we get equality (c.f. Remark 3.6).

If $i-1$ is not in the bottom row and $F_{i-1} = N^{-l}(\operatorname{im} N^{n-k-j})$ for r in the bottom row, then we know by construction that $i-1 = 2l+j$. By (4.1) we have $F_{i+1} = N^{-l-1}(\operatorname{im} N^{n-k-j})$, therefore by Theorem 3.7 we deduce $F_{i+2} = N^{-l-1}(\operatorname{im} N^{n-k-j-1})$. So we have

$$F'_i = NF_{i+2} = NN^{-l-1}(\operatorname{im} N^{n-k-j-1}) = N^{-l}(\operatorname{im} N^{n-k-j}),$$

where we get again equality by dimension calculations.

Thus, the conditions of Fung's theorem is satisfied for i in the top row of $p(w)$.

If F_k with $k > i+2$ and $k \in w_\wedge$ is dependent on F_l with $l \leq i-1$ or on $\operatorname{im} N^{n-k-l}$, then the conditions are satisfied by the same computations. If F_k is dependent on F_l with $l > i+2$, then the conditions are fulfilled by analogous considerations, since we get $NN^{-j}F_l = N^{-j}NF_l$ by dimension reasons.

Therefore the conditions of Fung's theorem are satisfied and we get $(F_1 \subset \cdots \subset F_{i-1} \subset NF_{i+2} \subset \cdots \subset NF_n) \in Y_{p(S)}$. Since this holds for every standard tableau, it holds in particular for $S(w)$.

Let $(F_1 \subset \cdots \subset F_{i-1} \subset F_{i+1} \subset \cdots \subset F_n) \in \widetilde{\mathcal{Y}}_w$ for w a row strict tableau. Then we have $(F_1 \subset \cdots \subset F_{i-1} \subset NF_{i+2} \subset \cdots \subset NF_n) \in \mathcal{Y}_{p(w)}$:
This assertion follows from Theorem 4.6 and [SW08, Theorem 15]. Indeed, consider an element $j \in (w_\wedge \cap S(w)_\vee) \setminus w_\times$. Then we have $F_j = F_j(w)$. Therefore, if $j \leq i-1$, we get $F'_j = F_j(p(w))$, and if $j \geq i+2$, we get

$$F'_{j-2} = NF_j = NF_j(w) = F_{j-2}(p(w)).$$

Furthermore, we have

$$p(w)_\wedge \cap S\big(p(w)\big)_\vee = p\big((w_\wedge \cap S(w)_\vee) \setminus w_\times\big)$$

Thus, the conditions of [SW08, Theorem 15] are satisfied and $(F_1 \subset \cdots \subset F_{i-1} \subset NF_{i+2} \subset \cdots \subset NF_n) \in \mathcal{Y}_{p(w)}$ follows.

Now we consider the map $\pi' : \mathcal{Y}_{p(w)} \to \widetilde{\mathcal{Y}}_w$ given by

$$(F'_1 \subset \cdots \subset F'_{n-2}) \mapsto \left(F'_1 \subset \cdots \subset F'_{i-1} \subset N^{-1}F'_{i-1} \subset \ldots N^{-1}F'_{n-2}\right).$$

Analogously to the above calculation one can compute that it is well-defined.

We have

$$\begin{aligned} &\pi\left(\pi'\left((F'_1 \subset \cdots \subset F'_{n-2})\right)\right) \\ &= \left(F'_1 \subset \cdots \subset F'_{i-1} \subset NN^{-1}F'_i \subset \cdots \subset NN^{-1}F'_{n-2}\right) \\ &= (F'_1 \subset \cdots \subset F'_{i-1} \subset F'_i \subset \cdots \subset F'_{n-2}) \end{aligned}$$

and

$$\begin{aligned} &\pi'\left(\pi\left((F_1 \subset \cdots \subset F_{i-1} \subset F_{i+1} \subset \cdots \subset F_n)\right)\right) \\ &= \left(F_1 \subset \cdots \subset F_{i-1} \subset N^{-1}F_{i-1} \subset N^{-1}NF_{i+2} \cdots \subset N^{-1}NF'_{n-2}\right) \\ &= (F_1 \subset \cdots \subset F_{i-1} \subset F_{i+1} \subset \cdots \subset F_n), \end{aligned}$$

since (4.1) holds as well as $F_j \subset N^{-1}NF_j$ and the subspaces have the same dimension for $j = i+2, \ldots, n$. Therefore, we get bijectivity.

π is a morphism of varieties:
We consider $\widetilde{Y}$ as subset of $Gr(1,\mathbb{C}^n) \times \cdots \times Gr(i-1,\mathbb{C}^n) \times Gr(i+1,\mathbb{C}^n) \times \cdots \times Gr(n,\mathbb{C}^n)$ given by all tuples $(F_1,\ldots,F_{i-1},F_{i+1},\ldots,F_n)$ such that $(F_1 \subset \cdots \subset F_{i-1} \subset F_{i+1} \subset \cdots \subset F_n)$ is N-invariant, where $Gr(j,\mathbb{C}^n)$ is the Grassmannian variety of j-dimensional subspaces in $\mathbb{C}^n$. Furthermore, we consider

$$\begin{aligned} \widehat{\pi} : \widetilde{Y} &\to \bigtimes_{i=1}^{n-2} Gr(i,\mathbb{C}^{n-2}) \\ (F_1,\ldots,F_{i-1},F_{i+1},\ldots,F_n) &\mapsto (F_1,\ldots,F_{i-1},NF_{i+2},\ldots,NF_n). \end{aligned}$$

As the dimensions fit by the above statements, $\widehat{\pi}$ is well-defined. Now we show that $\widehat{\pi}$ is a morphism of varieties.

Since $F_{i-1} = NF_{i+1}$, we have $\ker N \subset F_j$ for $j > i$ and $F_j \subset N\mathbb{C}^n$ for $j < i$. We define $\gamma : N\mathbb{C}^n \to \mathbb{C}^n/\ker N$ by $Ne_j \mapsto e_j + \ker N$, $Nf_j \mapsto f_j + \ker N$.

We write $\widehat{\pi}$ as composition of two maps which we show to be morphisms of varieties:

$$\begin{aligned} \psi : \widetilde{Y} &\to \bigtimes_{i=1}^{n-2} Gr\left(i,\mathbb{C}^n/\ker N\right) \\ (F_1 \subset \cdots \subset F_{i-1} \subset F_{i+1} \subset \cdots \subset F_n) &\mapsto \left(\gamma(F_1),\ldots,\gamma(F_{i-1}),F_{i+2}/kerN,\ldots,F_n/\ker N\right) \end{aligned}$$

and

$$\begin{aligned} \rho : \bigtimes_{i=1}^{n-2} Gr\left(i,\mathbb{C}^n/\ker N\right) &\to \bigtimes_{i=1}^{n-2} Gr\left(i,\mathbb{C}^{n-2}\right) \\ (F'_1,\ldots,F'_{n-2}) &\mapsto \left(\beta(F'_1),\ldots\beta(F'_{n-2})\right) \end{aligned}$$

for $\beta : \mathbb{C}^n/\ker N \to N\mathbb{C}^n = \mathbb{C}^{n-2}$.

At first, we show that

$$\begin{aligned}\vartheta : X_l := \{V \in Gr(l,\mathbb{C}^n) | \ker N \subset V\} &\to Gr(l-2, \mathbb{C}/\ker N)\\ V &\mapsto V/\ker N\end{aligned}$$

is a morphism of varieties:

For W a vector space the Plücker embedding, which identifies $Gr(r,W)$ with a closed subset of $\mathbb{P}(\bigwedge^r W)$, is given by $\langle w_1,\dots,w_r\rangle \mapsto w_1 \wedge \cdots \wedge w_r$ [FH04, 15.4]. We consider the Plücker embedding $X_l \to \mathbb{P}\left(\bigwedge^l \mathbb{C}^n\right)$ and $Gr(l-2,\mathbb{C}/\ker N) \to \mathbb{P}\left(\bigwedge^{l-2} \mathbb{C}^n/\ker N\right)$. Because of $\ker N = \langle e_1, f_1\rangle$, for $V \in X_l$ we have

$$V = \langle e_1, f_1, v_1, \dots v_{l-2}\rangle \text{ with } v_1,\dots,v_{l-2} \in \langle e_2,\dots,e_{n-k}, f_2,\dots,f_k\rangle .$$

Then the map inducing ϑ is given by

$$\begin{aligned}\widehat{\vartheta} : \mathbb{P}\left(\bigwedge^l \mathbb{C}^n\right) &\to \mathbb{P}\left(\bigwedge^{l-2} \mathbb{C}^n/\ker N\right)\\ \left[a_1 : \cdots : a_{\binom{n}{l}}\right] &\mapsto \left[a_1 : \cdots : a_{\binom{n-2}{l-2}}\right]\end{aligned}$$

for a fitting numbering of the bases of $\mathbb{P}\left(\bigwedge^l \mathbb{C}^n\right)$ and $\mathbb{P}\left(\bigwedge^{l-2} \mathbb{C}^n/\ker N\right)$, which are given by the wedges of the e_i, f_i or $e_i + \ker N$, $f_i + \ker N$, respectively. In particular, this map is given by homogeneous polynomials, and thus by [Mil08, Proposition 6.17] a morphism of varieties. Therefore, this also holds for ϑ as a restriction to closed subvarieties.

Again, with the Plücker embedding one can see that

$$\begin{aligned}\widehat{\gamma} : Gr(j,\mathbb{C}^n) &\to Gr(j,\mathbb{C}^n/\ker N)\\ F_j &\mapsto \gamma(F_j)\end{aligned}$$

is a morphism of varieties for $j < i$.

Consider the projections

$$\begin{aligned}&pr_j : \bigtimes_{i=1}^{n-2} Gr(i,\mathbb{C}^n/\ker N) \to Gr(j,\mathbb{C}^n/\ker N) \quad \text{and}\\ &pr'_j : Gr(1,\mathbb{C}^n) \times \cdots \times Gr(i-1,\mathbb{C}^n) \times Gr(i+1,\mathbb{C}^n) \times \cdots \times Gr(n,\mathbb{C}^n) \to Gr(j,\mathbb{C}^n)\end{aligned}$$

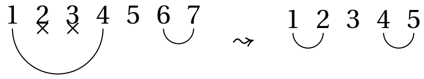
to the j-th factor. For showing that ψ is a morphism of varieties by the universal property of the product it suffices to show that $pr_j \circ \psi$ is a morphism of varieties. But we have $pr_j \circ \psi = \widehat{\gamma} \circ pr'_j$ for $j < i$ and $pr_j \circ \psi = \vartheta \circ pr'_j$ for $j > i$.

Since pr'_j is a morphism of varieties and $\widehat{\gamma}$ and ϑ are also morphisms of varieties, $pr_j \circ \psi$ is a morphism of varieties and thus ψ is one as well.

Analogously as above one can show that $\widehat{\beta} : Gr(j,\mathbb{C}^n/kerN) \to Gr(j,\mathbb{C}^{n-2})$ is a morphism of varieties. Again we have $pr''_j \circ \rho = \widehat{\beta} \circ pr'_j$ for $pr''_j : Gr(1,\mathbb{C}^{n-2}) \times \cdots \times Gr(n-2,\mathbb{C}^{n-2}) \to Gr(j,\mathbb{C}^{n-2})$ the projections. Therefore, $pr''_j \circ \rho$ is a morphism of varieties and so ρ is also one.

Altogether, $\widehat{\pi} = \rho \circ \psi$ is a morphism of varieties and thus π, being the restriction to closed subsets, is one as well. $\square$

Remark 4.9. The above theorem gives a connection between the generalised irreducible components of the Springer fibres and those of the Spaltenstein varieties. Visualising this connection in cup diagrams leads to simply forgetting the $\times\times$.



In the following section we discuss further connections using a graphical tool which we call the dependence graph.

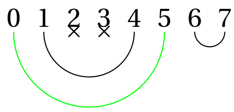
5 Dependence graphs

5.1 Dependence graphs for standard tableaux

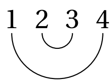
We start this section by defining extended cup diagrams and dependence graphs for standard tableaux. Then we introduce more notation and provide some technical tools. Finally we use these to show that the structure of an irreducible component is given by the conditions of the corresponding dependence graph.

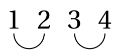
Definition 5.1. Let S be a standard tableau of type $(i_1,\dots,i_m)$. The *extended cup diagram for S*, $eC(S)$, is defined as follows: We expand the weight sequence from Definition 4.3 by adding $n-2k$ $\vee$'s on the left, i.e. $\mathbf{a} = \underbrace{\vee\dots\vee}_{n-2k} a_1 a_2 a_3 \dots a_n$. Then we connect the $\vee\wedge$-pairs as before. If a cup is starting at one of the newly added $\vee$'s, we colour it green.

Example 5.2. $n = 7, k = 3, n-2k = 1$, $S = \begin{array}{|c|c|c|c|}\hline 7&5&4&3\\\hline 6&3&1\\\cline{1-3}\end{array}$, $\mathbf{a} = \vee\vee\times\times\wedge\wedge\vee\wedge$

$eC(S) =$ 

Example 5.3. If $n = 2k$, the extended cup diagrams coincide with the cup diagrams, for example $n = 4, k = 2$. Then we have the following two standard tableaux of type $(1,2,3,4)$: $S_1 = \begin{array}{|c|c|}\hline 4&3\\\hline 2&1\\\hline\end{array}$ and $S_2 = \begin{array}{|c|c|}\hline 4&2\\\hline 3&1\\\hline\end{array}$.

$eC(S_1) =$ 

$eC(S_2) =$ 

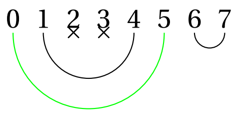
Definition 5.4. Let S be a standard tableau of type $(i_1,\dots,i_m)$. The *dependence graph* for S, $depG(S)$, is defined as follows:

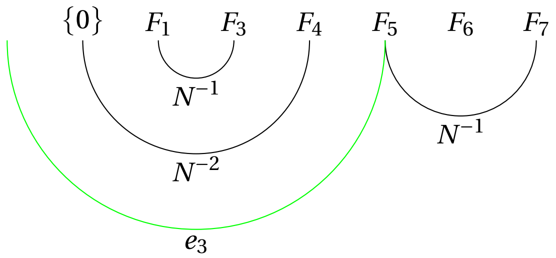
We have $m+(n-2k)+1$ given nodes, numbered $-(n-2k)$ to 0 and i_1 to i_m. We label the nodes with F_j for $j \in \{i_1,\dots,i_m\}$ and with $\{0\}$ for the node 0; the remaining nodes are left unlabelled.

If i_s is labelled with $\times$ in the extended cup diagram, then in the dependence graph we connect i_s-2 and i_s and label the resulting arc with N^{-1}.

Now, if $i < j$ are connected in the extended cup diagram, then in the dependence graph we connect the nodes $i-1$ and j by an arc of the same colour. We label the black arcs with N^{-l} for $l = \frac{1}{2}(j-(i-1))$ and the green ones with e_l.

Remark 5.5. Note that l always is an integer. We constructed the extended cup diagram by connecting adjacent nodes after an even number in between them is deleted. Thus i and j have different parity and $i-1$ and j have the same parity.

Example 5.6. $S = \begin{array}{|c|c|c|c|}\hline 7 & 5 & 4 & 3 \\ \hline 6 & 3 & 1 & \\ \cline{1-3}\end{array}$ $\quad eC(S) =$ 

$depG(S) =$ 

Definition 5.7. Let S be a standard tableau of type $(i_1, \ldots, i_m)$. A flag $(F_{i_1} \subset \cdots \subset F_{i_m})$ *satisfies the conditions of* $depG(S)$ if we have

1. if the node labelled F_i $(i > 0)$ is connected to a node labelled F_j with $i < j$ via a black arc labelled N^{-l}, then $F_j = N^{-l}F_i$
2. if the node labelled F_i is the endpoint of a green arc labelled e_l, then $F_i = F_{i-1} + \langle e_l \rangle$

The goal in this section is to prove the following theorem:

Theorem 5.8. *Let S be a standard tableau of type $(i_1, \ldots, i_m)$. Then the irreducible components $\widetilde{Y}_S$ consist of all N-invariant flags satisfying the conditions of the dependence graph for S.*

The space F_j, $j > 0$, is independent in $\widetilde{Y}_S$ if and only if the node labelled F_j is the node at the left end of a black connected component of the dependence graph for S, where a node without arcs is also a component.

Before we can start with the proof of this theorem, we need more notation and a whole bunch of statements about dependence graphs.

Definition 5.9. Let B be an arc in $depG(S)$. Then we denote by $s(B)$ the number of the left endpoint of the arc and by $t(B)$ the number of the right endpoint. (Here, by number we mean the number of the node as defined in Definition 5.4 and not its position.) We define the width $b(B)$ via $b(B) = \frac{1}{2}(t(B) - s(B))$.

An arc B' *is nested inside* B if we have $s(B) \le s(B') < t(B') \le t(B)$. Note that B is nested inside B. An *arc sequence* from a to b is given by arcs $B_1, \ldots, B_r$ with $s(B_1) = a$, $t(B_r) = b$

and $t(B_i) = s(B_{i+1})$ for $i = 1, \dots, r-1$. For G a green arc let $g(G)$ be the number of green arcs nested inside G.

If B is an arc in $depG(S)$, we denote by $\overset{\smile}{\mathrm{B}}$ the cup in $eC(S)$ from which it arises; here we allow $\overset{\smile}{\mathrm{B}}$ to be $\times\times$. Also in the extended cup diagram, we denote by $s(\overset{\smile}{\mathrm{B}})$ the left end of a cup and by $t(\overset{\smile}{\mathrm{B}})$ the right end, where we treat $\times\times$ as a cup.

Remark 5.10. In this notation, by the definition of the dependence graph the labelling of an arc B in $depG(S)$ is given by $N^{-b(B)}$ or $e_{b(B)}$, respectively.

Furthermore, we have $t(B) = t(\overset{\smile}{\mathrm{B}})$ and $s(B) = s(\overset{\smile}{\mathrm{B}}) - 1$, and B has the same colour as $\overset{\smile}{\mathrm{B}}$.

Lemma 5.11. *a) Arcs in the dependence graph do not intersect.*

b) We have $s(B) \geq 0$ for all black arcs in the dependence graph.

c) We have $s(G) \leq -1$ for all green arcs in the dependence graph.

d) We have $s(G) = -g(G)$ for all green arcs in the dependence graph.

e) We have $b(G) = g(G) + \frac{1}{2}(t(G) - g(G))$ for all green arcs in the dependence graph.

f) We have $k \in S_\wedge$ if and only if there is an arc B with $k = t(B)$ in the dependence graph.

g) For B an arc in the dependence graph, we have $b(B) > 1$ if and only if $t(B) - 1 \in S_\wedge$.

Proof. a) By construction, arcs in the cup diagram do not intersect and no left endpoint is a right endpoint of another cup. So they do not intersect, if we move all left endpoints one to the left. In addition they do not intersect with arcs coming from a $\times\times$, since the construction proceeds as if there is a cup connecting the $\times$.

b) Let B be a black arc in the dependence graph. Then by construction we have $s(\overset{\smile}{\mathrm{B}}) \geq 1$, therefore $s(B) \geq 0$.

c) Let G be a green arc in the dependence graph. By construction for the green arc $\overset{\smile}{\mathrm{G}}$ we have $s(\overset{\smile}{\mathrm{G}}) \leq 0$, thus $s(G) \leq -1$.

d) Let G be a green arc. By construction, the nodes with the numbers $-1, -2, \dots$ in the dependence graph are only starting points of green arcs, and there are no gaps where no green arc is starting. Thus, there are as many arcs nested inside G as $-s(G)$.

e) By d) we have:

$$b(G) = \frac{1}{2}(t(G) - s(G)) = \frac{1}{2}(t(G) + g(G)) = g(G) + \frac{1}{2}(t(G) - g(G))$$

f) We have $k \in S_\wedge$ if and only if k is the right end of a cup in the cup diagram or it is a double entry in the tableau. On the other hand, by construction we have $k = t(B)$ for B an arc in the dependence graph if and only if one of these cases holds.

g) By f) $t(B) \in S_\wedge$. Assume $t(B)-1 \notin S_\wedge$. For this there are two possibilities: either $t(B)-1 \in S_\vee$ and it is not a double entry or $t(B)-1 \notin S$ and $t(B)$ is a double entry. The first possibility is equivalent to the fact that there is a cup in the cup diagram between $t(B)-1$ and $t(B)$. Thus, both cases are equivalent to the fact that there is an arc between $t(B)-2$ and $t(B)$ in the dependence graph. So $t(B)-1 \notin S_\wedge$ is equivalent to $b(B) = 1$.

□

Remark 5.12. The nodes in $depG(S)$ at the left end of a black connected component of the dependence graph for S coincide with the nodes that are at the left end of a black cup in the extended cup diagram.

This holds, because k is the left end of a connected component if and only if there is no arc B in $depG(S)$ such that $k = t(B)$. By Lemma 5.11 f) this is equivalent to $k \notin S_\wedge$, i.e. $k \in S_\vee$ and k is not a double entry, which means that k is the left end of a cup.

Therefore, by Theorem 5.8 the number of independents in $\widetilde{Y}_S$ is the same as the number of black cups in $eC(S)$.

Proposition 5.13. *a) In the dependence graph there are only black arcs nested inside black arcs.*

b) Let B be an arc in the dependence graph with $b(B) > 1$. Then there is an arc B' with $s(B') > s(B)$ and $t(B') = t(B)-1$.

c) Let B be an arc in the dependence graph with $b(B) > 2$ and let B' be a black arc nested inside B which is not nested inside any other arc which is nested inside B and, in addition, satisfies $s(B') \neq t(G)$ for all green arcs G, $s(B') > 0$ and $s(B') > s(B)+1$. Then there is a black arc B'' with $s(B'') \geq 0$ and $t(B'') = s(B')$.

d) Let B be a black arc with width $b(B) > 1$. Then there is a black arc sequence from $s(B)+1$ to $t(B)-1$.

e) Let B be a green arc with $b(B) > 1$. If there is no green arc nested inside B, then there is an arc sequence from 0 to $t(B)-1$. If there is a green arc nested inside B, then there is an arc sequence from the rightmost endpoint of the green arcs nested inside B to $t(B)-1$.

f) The number of arcs nested inside B is equal to the width $b(B)$.

Proof. a) Let B be a black arc in the dependence graph and let G be a green one. Then by Lemma 5.11 b), c) we have $s(G) < s(B)$, which means by definition that G is not nested inside B.

b) Let B be an arc with $b(B) > 1$ in the dependence graph. By Lemma 5.11 g) we have $t(B)-1 \in S_\wedge$, so by 5.11 f) we have $t(B)-1 = t(B')$ for an arc B'. Since by 5.11 a) the arcs do not intersect, we have $s(B') > s(B)$ and therefore B' is nested inside B.

c) Consider $s(B')$. If $s(B') \in S_\wedge$, then by Lemma 5.11 f) there is an arc B'' with $t(B'') = s(B')$ which is nested inside B because of Lemma 5.11 a). Since no point is endpoint of two arcs, by assumption B'' is not green, hence it is black and $s(B'') \geq 0$.

   If $s(B') \in S_\vee$ and not a double entry, then by construction in $eC(S)$ there is a cup C with $s(C) = s(B')$ and $t(C) > t(B')$, since the cups do not intersect. Because of $s(B') > s(B)+1$ since no point is starting point of two arcs, in the dependence graph there is an arc $A \neq B$ with $\overset{\frown}{A} = C$ inside of which B' is nested. This is a contradiction.

   The case where $s(B') \notin S$ cannot appear, since such nodes are by definition not in the dependence graph.

d) By b) and a there is a black arc B' with $s(B') > s(B)$ and $t(B') = t(B)-1$. If B' does not form the desired arc sequence, we apply c) as long as $s(B') > s(B)+1$. Since by a) there are only black arcs nested inside B and by Lemma 5.11 b) we have $s(B) \geq 0$, we can apply c) as long as $s(B') > s(B)+1$. If we do not have $s(B') > s(B)+1$ any more, we have $s(B') = s(B)+1$, because the arcs do not intersect and do not have the same starting point, and we have found the desired arc sequence.

e) By b) there is an arc B' with $s(B') > s(B)$ and $t(B') = t(B)-1$. If it satisfies the conditions, we apply c) as long as possible, otherwise it is green. Since by Lemma 5.11 c) we have $s(B) \leq -1$, we can apply c) until $s(B') = t(G)$ for a green arc G or $s(B') = 0$.

f) First assume that B is a black arc and use induction on $b(B)$:
   For $b(B) = 1$ there cannot be another arc nested inside B. For $b(B) \geq 2$ by d) there is an arc sequence $B_1, \ldots, B_l$ from $s(B)+1$ to $t(B)-1$. We have $b(B)-1 = \sum b(B_i)$ and by induction we know $b(B_i) = \#\{\text{arcs nested inside } B_i\}$. Therefore, $b(B) = 1 + \sum b(B_i) = \#\{\text{arcs nested inside } B\}$ follows.

   Now let B be a green arc. We use induction on $g(B)$: For $g(B) = 1$ it follows that $s(B) = -1$, since only green arcs start on negative numbers and there are no gaps by construction. By e) there is a black arc sequence $B_1, \ldots, B_r$ from $0 = s(B)+1$ to $t(B)-1$. We have $b(B)-1 = \sum b(B_i)$ and we showed above that $b(B_i) = \#\{\text{arcs nested inside } B_i\}$. Therefore, $b(B) = 1 + \sum b(B_i) = \#\{\text{arcs nested inside } B\}$ follows.

   Let $g(B) \geq 2$ and let M be a green arc such that

$$t(M) = \max\{t(B') | B' \neq B \text{ is a green arc nested inside B}\}.$$

   By e) there is a black arc sequence from $t(M)$ to $t(B)-1$. By definition of M we have $s(M) = s(B)+1$. Therefore, there is an arc sequence from $s(B)+1$ to $t(B)-1$, and for all these arcs the assumption holds. Thus, $b(B) = \#\{\text{arcs nested inside } B\}$ follows as above.

□

Proof of Theorem 5.8. Let $F_\bullet \in \widetilde{Y}$. We want to show: We have $F_\bullet \in \widetilde{Y}_S$ if and only if $F_\bullet$ is N-invariant and satisfies the conditions of the dependence graph for S. By Theorem 3.7 the former is equivalent to $F_\bullet$ satisfying the conditions listed in Theorem 3.7. Since the first condition of the ones listed in Theorem 3.7 coincides with the N-invariance, we only analyse the other conditions. These conditions are only conditions for $k \in S_\wedge$, which is by Lemma 5.11 f) equivalent to $k = t(B)$ for B some arc in $depG(S)$.

Because of this and the fact that the dependence graph only gives conditions for $k = t(B)$ as well, we consider all $t(B)$ for B an arc in $depG(S)$. We start with the case that B is a black arc and proceed by induction on $b(B)$.

Let $b(B) = 1$. By Lemma 5.11 g) we have $s(\breve{B}) = s(B) + 1 = t(B) - 1 \in S_\vee$ or $t(B) - 1 \notin S$ and $t(B)$ is a double entry. If $t(B)$ is a double entry, we have $F_{t(B)} = N^{-1} F_{t(B)-2} = N^{-b(B)} F_{s(B)}$ by Remark 3.6. Otherwise, by Theorem 3.7 we have $F_{t(B)} = N^{-1} F_{t(B)-2} = N^{-b(B)} F_{s(B)}$. Thus the conditions of Theorem 3.7 and those of the dependence graph are the same.

For $b(B) > 1$, by Lemma 5.11 g) we have $t(B) - 1 \in S_\wedge$. By Proposition 5.13 d) there is a black arc sequence from $s(B) + 1$ to $t(B) - 1$. Since the width of the arcs nested inside B is smaller than the width of B, the conditions of the B_i of the arc sequence agree with the conditions of Theorem 3.7 by induction. Thus, we have $F_{t(B_i)} = N^{-b(B_i)} F_{s(B_i)}$ and $F_{t(B)-1} = N^{-b(B)+1} F_{s(B)+1}$, since $b(B) - 1 = \sum b(B_i)$. An arc A with $t(A) = s(B) + 1$ would intersect B, thus $F_{s(B)+1}$ is independent. Therefore, the conditions of Theorem 3.7 imply $F_{t(B)} = N^{-b(B)} F_{s(B)}$, which is equivalent to the conditions of the dependence graph.

For G a green arc consider

$$\begin{aligned} F_{t(G)-1} &= N^{-\frac{1}{2}\left(t(G)-g(G)\right)}\left(\operatorname{im} N^{n-k-g(G)+1}\right) \\ F_{t(G)} &= N^{-\frac{1}{2}\left(t(G)-g(G)\right)}\left(\operatorname{im} N^{n-k-g(G)}\right) \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} F_{t(G)-1} &= N^{-\frac{1}{2}\left(t(G)-g(G)\right)}\left(\operatorname{im} N^{n-k-g(G)+1}\right) \\ F_{t(G)} &= F_{t(G)-1} + \left\langle e_{b(G)} \right\rangle. \end{aligned} \tag{5.2}$$

(5.1) and (5.2) are equivalent:

We have $g(G) \leq n - 2k$, because there are $n - k$ endpoints of arcs and k of them are connected to black starting points, so there are $n - 2k$ endpoints left for green arcs and the double entries decrease these numbers even more. For $a \leq n - 2k$ we have $\operatorname{im} N^{n-k-a} = \langle e_1, \ldots, e_a \rangle$. Thus,

$$\begin{aligned} \operatorname{im} N^{n-k-g(G)+1} &= \left\langle e_1, \ldots, e_{g(G)-1} \right\rangle \\ \operatorname{im} N^{n-k-g(G)} &= \left\langle e_1, \ldots, e_{g(G)-1} \right\rangle + \left\langle e_{g(G)} \right\rangle \end{aligned}$$

$$\begin{aligned} &N^{-\frac{1}{2}\left(t(G)-g(G)\right)}\left(\operatorname{im} N^{n-k-g(G)}\right) \\ &\qquad = N^{-\frac{1}{2}\left(t(G)-g(G)\right)}\left(\left\langle e_1, \ldots, e_{g(G)-1} \right\rangle\right) + \left\langle e_{g(G)+\frac{1}{2}(t(G)-g(G))} \right\rangle \\ &\qquad = N^{-\frac{1}{2}\left(t(G)-g(G)\right)}\left(\left\langle e_1, \ldots, e_{g(G)-1} \right\rangle\right) + \left\langle e_{b(G)} \right\rangle, \end{aligned}$$

where the last equation holds because of Lemma 5.11 e). Therefore, under the precondition $F_{t(G)-1} = N^{-\frac{1}{2}\left(t(G)-g(G)\right)}\left(\operatorname{im} N^{n-k-g(G)+1}\right)$, we conclude that the equation $F_{t(G)} = N^{-\frac{1}{2}\left(t(G)-g(G)\right)}\left(\operatorname{im} N^{n-k-g(G)}\right)$ is equivalent to the equation $F_{t(G)} = F_{t(G)-1} + \left\langle e_{b(G)} \right\rangle$.

Now we show by induction on $g(G)$, that for a green arc G the conditions for $t(G)$ of Theorem 3.7 agree with the conditions of the dependence graph. For this we use that we already know that the conditions coincide on black arcs by the first part of the proof.

For $g(G) = 1$ we have either $s(G) = -1$, $t(G) = 1$ (since this is the only possibility for $b(G) = 1$) or by Proposition 5.13 e) there is a black arc sequence from 0 to $t(G) - 1$.

In the first case, by Lemma 5.11 f) we have $1 = t(G) \in S_\wedge$ and by convention we have $t(G) - 1 = 0 \in S_\wedge$ and $F_{t(G)-1} = F_0 = N^{-0}\left(\operatorname{im} N^{n-k}\right)$. Therefore, by the conditions listed in Theorem 3.7 we have

$$F_{t(G)} = N^{-0}(\operatorname{im} N^{n-k-1}) = N^{-\frac{1}{2}(t(G)-g(G))}(\operatorname{im} N^{n-k-g(G)}),$$

which by the previous considerations agrees with $F_{t(G)} = F_{t(G)-1} + \langle e_{b(G)} \rangle$.

In the second case, we have $F_{t(G)-1} = N^{-\frac{1}{2}\left(t(G)-1\right)}\left(\operatorname{im} N^{n-k}\right)$ by the first part of the proof, because the product of the labels of the arcs nested inside is exactly $N^{-\frac{1}{2}\left(t(G)-1\right)}$ and $F_0 = \operatorname{im} N^{n-k}$. Therefore, by Lemma 5.11 f), g) we get $F_{t(G)} = N^{-\frac{1}{2}\left(t(G)-1\right)}\left(\operatorname{im} N^{n-k-1}\right)$ by the conditions listed in Theorem 3.7. By the discussion above this again agrees with the conditions of the dependence graph.

For $g(G) \geq 2$ let M be a green arc with

$$t(M) = \max\{t(B) | B \neq G \text{ is a green arc nested inside } G\}.$$

Then we have $g(G) = g(M) + 1$. By Proposition 5.13 e) there is a black arc sequence from $t(M)$ to $t(G) - 1$. By induction we have

$$F_{t(M)} = N^{-\frac{1}{2}\left(t(M)-g(M)\right)}\left(\operatorname{im} N^{n-k-g(M)}\right)$$

in the dependence graph as well as given by the conditions listed in Theorem 3.7. Thus,

$$\begin{aligned} F_{t(G)-1} &= N^{-\left(\frac{1}{2}\left(t(M)-g(M)\right)+\frac{1}{2}\left(t(G)-1-t(M)\right)\right)}\left(\operatorname{im} N^{n-k-g(M)}\right) \\ &= N^{-\frac{1}{2}\left(t(G)-g(M)-1\right)}\left(\operatorname{im} N^{n-k-g(M)}\right) = N^{-\frac{1}{2}\left(t(G)-g(G)\right)}\left(\operatorname{im} N^{n-k-g(M)}\right). \end{aligned}$$

Therefore by Lemma 5.11 f), g) we conclude

$$F_{t(G)} = N^{-\frac{1}{2}\left(t(G)-g(G)\right)}\left(\operatorname{im} N^{n-k-g(G)}\right)$$

from the conditions listed in Theorem 3.7. Again, this agrees with the conditions of the dependence graph.

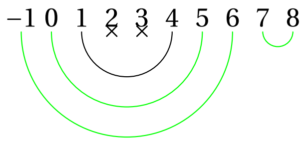
□

5.2 Dependence graphs for row strict tableaux

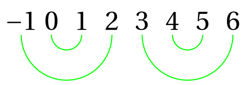
In this chapter we define extended cup diagrams and dependence graphs for row strict tableaux generalising the definitions of the last section. Then we prove that the structure of a generalised irreducible component is given by the conditions of the associated dependence graph. At the end we define a morphism from a generalised irreducible component of a Springer fibre to an irreducible component using dependence graphs.

Definition 5.14. Let w be a row strict tableau of type $(i_1,\ldots,i_m)$. The *extended cup diagram* for w, $eC(w)$, is defined as follows: We add $n-k$ $\vee$'s on the left and k $\wedge$'s on the right of the weight sequence, i.e. $\mathbf{a} = \underbrace{\vee\ldots\vee}_{n-k} a_1 a_2 a_3 \ldots a_n \underbrace{\wedge\ldots\wedge}_{k}$. Then we connect $\vee\wedge$ as usual until all of the nodes $1,\ldots,n$ are connected. After that we delete the remaining ones of the added $\vee$'s and $\wedge$'s. If an arc is starting at one of the newly added $\vee$'s or ending at one of the newly added $\wedge$'s, we colour it green.

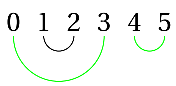
Example 5.15. $w = \begin{array}{|c|c|c|c|}\hline 6&5&4&3\\\hline 7&3&1\\\cline{1-3}\end{array}$, $\mathbf{a} = \vee\vee\vee\vee\vee\times\times\wedge\wedge\wedge\vee\wedge\wedge\wedge$.

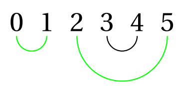
$eC(w) =$ 

Example 5.16. If $n = 2k$, the extended cup diagrams for a row strict tableau which is not a standard tableau do not coincide with the normal cup diagrams:
Let $n = 4, k = 2$. Then, in addition to the 2 standard tableaux of type $(1,2,3,4)$, there are the following row strict tableaux of type $(1,2,3,4)$:
$w_1 = \begin{array}{|c|c|}\hline 2&1\\\hline 4&3\\\hline\end{array}$, $w_2 = \begin{array}{|c|c|}\hline 3&1\\\hline 4&2\\\hline\end{array}$, $w_3 = \begin{array}{|c|c|}\hline 3&2\\\hline 4&1\\\hline\end{array}$ and $w_4 = \begin{array}{|c|c|}\hline 4&1\\\hline 3&2\\\hline\end{array}$.

$eC(w_1) =$ 

$eC(w_2) =$ 

$eC(w_3) =$ 

$eC(w_4) =$ 

Definition 5.17. Let w be a row strict tableau of type $(i_1,\ldots,i_m)$. The *dependence graph for* w, *depG*(w), is defined as follows: We have m given nodes, numbered i_1 to i_m. To the left of these we add one node more than there are nodes in $eC(w)$ to the left of the node 1 and number the new nodes by $\ldots,-1,0$. Analogously, to the right we add one node more than there are nodes in $eC(w)$ to the right of the node n and number the new nodes by $n+1, n+2, \ldots$. We label the nodes with F_j for $j \in \{i_1,\ldots,i_m\}$ and with $\{0\}$ for the node 0; the remaining nodes are left unlabelled.

If i_s is labelled with $\times$ in the extended cup diagram, then we connect $i_s - 2$ and i_s and label the resulting arc with N^{-1}.

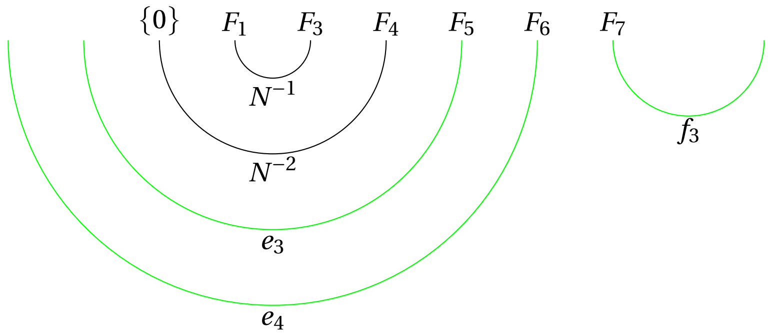
Now, if i and j with $i < j \leq n$ are connected in the extended cup diagram, then in the dependence graph we connect the nodes $i-1$ and j by an arc of the same colour. We label the black arcs with N^{-l} and the green ones with e_l, where $l = \frac{1}{2}(j-(i-1))$.

If i and j with $i \leq n < j$ are connected in the extended cup diagram, then we connect i and $j+1$ with a green arc and label it with $f_{k+1-b(B)}$.

We use the notation of Definition 5.9 in this case as well.

Example 5.18. $w = \begin{array}{|c|c|c|c|}\hline 6 & 5 & 4 & 3 \\ \hline 7 & 3 & 1 \\ \cline{1-3}\end{array}$

$depG(w) =$



Definition 5.19. Let w be a row strict tableau of type $(i_1, \ldots, i_m)$. A flag $(F_{i_1} \subset \cdots \subset F_{i_m})$ *satisfies the conditions of* $depG(w)$ if we have

1. if the node labelled F_i $(i > 0)$ is connected to a node labelled F_j with $i < j$ via a black arc labelled N^{-l}, then $F_j = N^{-l} F_i$
2. if the node labelled F_i is the endpoint of a green arc labelled e_l, then $F_i = F_{i-1} + \langle e_l \rangle$
3. if the node labelled F_i is the starting point of a green arc labelled f_l, then $F_i = F_{i-1} + \langle f_l \rangle$

The next theorem connects, similarly to the one before, the dependence graphs with the generalised irreducible components.

Theorem 5.20. *Let w be a row strict tableau of type $(i_1, \ldots, i_m)$.*

Then $\widetilde{\mathfrak{Y}}_w$ consist of all N-invariant flags satisfying the conditions of the dependence graph for w.

The space F_j, $j > 0$, is independent in $\widetilde{\mathfrak{Y}}_w$ if and only if the node labelled F_j is the node at the left end in a black connected component of the dependence graph for w.

Again, before proving the theorem, we make some remarks and prove some lemmata that help us understand the dependence graphs for row strict tableaux more deeply.

Remark 5.21. For all arcs in the dependence graph for a row strict tableau which are black or to the left of the leftmost green arc G with $t(G) > n$ the results of Proposition 5.13 and Lemma 5.11 hold.
Indeed, in the proof of these theorems we only considered the way the arcs in the dependence graph are constructed out of the cups in the extended cup diagram. For the arcs we mentioned above these construction rules are the same as those for the construction of the dependence graph for a standard tableau.

Remark 5.22. The nodes at the left end of a black connected component of the dependence graph for w coincide with the left ends of black arcs in $eC(w)$. This is true because by Remark 5.21 we know that Lemma 5.11 f) also holds in this case, and the assertion follows in the same way as in Remark 5.12. Therefore, by Theorem 5.20 the number of independents in $\widetilde{\mathfrak{Y}}_w$ is the same as the number of black cups in $eC(w)$.

Lemma 5.23. *Again, let $w_\times$ be the set of double entries and*

$$W_1 := (w_\wedge \cap S(w)_\vee) \setminus w_\times, \qquad W_2 := (w_\vee \cap S(w)_\wedge) \setminus w_\times.$$

a) All of the arcs in the dependence graph for $S(w)$ with $s(B)+1 \in W_1$, $t(B) \in W_2$ are nested.

b) Let B be a cup in $eC(S(w))$ with $s(B) \in W_1$, $t(B) \in W_2$. Then we have

$$\#\{c \in W_1 | c \leq s(B)\} = \#\{c \in W_2 | c \geq s(B)\}.$$

c) Let G be a green arc in $\mathrm{depG}(w)$ *with $t(G) \in W_1$, then we have $t(H) \in W_1$ for all green arcs H under which G is nested.*

d) Let B be a black arc in the dependence graph for $S(w)$ with $t(B) \in W_2$. Then there are green arcs G and A in $\mathrm{depG}(w)$ *with $t(G) = s(B)+1$ and $s(A) = t(B)$.*

e) B is a black arc in $\mathrm{depG}(w)$ *if and only if B is a black arc in* $\mathrm{depG}(S(w))$ *and $s(B)+1 \notin W_1$.*

f) The green arcs in $\mathrm{depG}(S(w))$ *are exactly the green arcs G in* $\mathrm{depG}(w)$ *with $t(G) <$* $\min\{b | b \in W_1\}$.

Proof. To form $C(w)$ by Definition 4.3 we first connect all $\vee\wedge$ and then all $\wedge\vee$. We call the cups which arise from $\wedge\vee$ bogus cups. By definition, $S(w)$ is determined by $C(w) = C(S(w))$. Since there are no bogus cups in $C(S(w))$, W_1 is the set of left endpoints of bogus cups and W_2 is the set of right endpoints of bogus cups. This is true, because the left endpoints of bogus cups are exactly those which are in $w_\wedge$ and in $S(w)_\vee$ and are no double entry; analogously for right endpoints.

If $n-k > k$, there are $\wedge$s left after creating the bogus cups, which are not connected to other points. These are at the left of all bogus cups because otherwise they would have been connected in another way. We call this assertion the left-assertion.

Therefore, $eC(w)$ arises from $C(w)$ by at first connecting all the remaining $\wedge$s to the left, then deleting all bogus cups and then connecting all $\wedge$s to the left and all $\vee$s to the right. From this d) follows. This procedure is possible, since all the bogus cups are nested inside each other because otherwise we would have connected them in another way.

With this construction and the left-assertion we get c).

$eC(S(w))$ arises from $C(S(w))$ by connecting the remaining $\wedge$s to the left.

If we now compare how $eC(S(w))$ and $eC(w)$ arise from $C(S(w)) = C(w)$, then we see that the black arcs in $eC(w)$ are exactly the black arcs in $eC(S(w))$ with $s(\mathrm{B}) \notin W_1$. From this we get e). For $c := \min\{b | b \in W_1\}$ we see by this comparison, that the green arcs in $eC(S(w))$ are exactly the green arcs G in $eC(w)$ with $t(\mathrm{G}) < c$. Therefore, we get f).

Since W_1 is the set of left endpoints of bogus cups and W_2 is the set of right ones, we particularly have $\#W_1 = \#W_2$. Since all bogus cups are nested inside each other, we even have

$$\#\{c \in W_1 | c \leq s(B)\} = \#\{c \in W_2 | c \geq s(B)\}$$

for B an arc in the cup diagram for $S(w)$ with $s(B) \in W_1$, $t(B) \in W_2$. Thus, b) holds.

Since all bogus cups are nested inside each other and by passing to the dependence graph only their left endpoints are shifted by one to the left, we have a). □

Definition 5.24. A green arc G in the dependence graph is called *left green arc* if $s(G) < 0$ and *right green arc* otherwise.

Lemma 5.25. *Let W_1 and W_2 be defined as in the preceding lemma.*

a) Let B be an arc in depG(w) *with $t(B) \in W_1$. Then we have*

$$\#\{b \in w_\wedge | b \le t(B)\} = b(B) \text{ and } \#\{b \in w_\vee | b \le t(B)\} = t(B) - b(B).$$

b) Let G, M be left green arcs in depG(w) *with $g(G) = g(M) + 1$. We have*

$$\frac{1}{2}\big(t(G) - 1 - t(M)\big) = b(G) - 1 - b(M) \quad \text{and}$$
$$b(M) - g(M) + b(G) - 1 - b(M) = b(G) + s(G) = t(G) - b(G).$$

c) Let B be a black arc in depG$(S(w))$ *with $t(B) \in W_2$, let G be a green arc in* depG(w) *with $t(G) = s(B) + 1$ and let A be a green arc in* depG(w) *with $s(A) = t(B)$. Then we have $g(G) - (n - 2k) = g(A)$.*

d) Under the same conditions as in c) we have

$$t(G) - b(G) + b(B) = k + 1 - b(A).$$

e) Let B be a black arc in depG$(S(w))$ *with $t(B) \in W_2$ and G a green arc in* depG(w) *with $t(G) = s(B) + 1$. Furthermore, let C be a black arc in* depG$(S(w))$ *with $t(C) < t(B)$, $t(C) \in W_2$ and $t(C)$ maximal with this properties, and let G' be a green arc in* depG(w) *with $t(G') = s(C) + 1$. We have*

$$b(G') + b(C) - 1 + \frac{1}{2}\big(t(B) - 1 - t(C)\big) = b(G) - b(B) + 1 \quad \text{and}$$
$$t(G') - b(G') + b(C) + \frac{1}{2}\big(t(B) - t(C) + 1\big) = t(G) - b(G) + b(B) - 1.$$

Proof. a) Let B be an arc in *depG*(w) with $t(B) \in W_1$. We have

$$\#\{b \in w_\wedge | b \le t(B)\} = \#\{\text{arcs nested inside } B\} = b(B),$$

where the first equation holds since each $b \in w_\wedge$ is endpoint of an arc in *depG*(w), and the second equation holds by Remark 5.21 and Proposition 5.13 f).

Moreover, we have

$$\begin{aligned}
&\#\{b \in w_\vee | b \le t(B)\} \\
&= \big(\#\{1, \dots, t(B)\} - \#\{b \in w_\times | b \le t(B)\}\big) \\
&\quad - \big(\#\{b \in w_\wedge | b \le t(B)\} - \#\{b \in w_\times | b \le t(B)\}\big) \\
&= t(B) - b(B),
\end{aligned}$$

since the double entries appear in $w_\wedge$ and $w_\vee$ and the remaining entries are exactly in one of the two sets.

b) By Remark 5.21 Lemma 5.11 holds. The equation from Lemma 5.11 e) is equivalent to $t(G)+g(G)=2b(G)$. Thus, we have

$$\begin{aligned}
b(G)-1-b(M) &= \frac{1}{2}\big(2b(G)-2-2b(M)\big)\\
&= \frac{1}{2}\big(t(G)+g(G)-2-t(M)-g(M)\big)\\
&= \frac{1}{2}\big(t(G)-1-t(M)+g(G)-1-g(M)\big)\\
&= \frac{1}{2}\big(t(G)-1-t(M)\big).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
b(M)-g(M)+b(G)-1-b(M) &= b(G)-g(M)-1\\
&= b(G)-g(G)\\
&= t(G)-b(G),
\end{aligned}$$

because the equation from Lemma 5.11 e) is equivalent to $b(G)-g(G)=t(G)-b(G)$.

c) By Lemma 5.23 c) all of the green arcs B' with $t(B')\notin W_1$ are nested inside G. Since there are exactly $n-2k$ of them, we have

$$g(G)-(n-2k)=\#\{c\in W_1 | c\le t(G)\}.$$

Moreover, we have $g(A)=\#\{c\in W_2 | c\ge s(A)\}$, because there is an arc starting at each of these c which is nested inside A. By Lemma 5.23 b) we have

$$\#\left\{c\in W_1 | c\le s(\overset{\smile}{\mathrm{B}})\right\}=\#\left\{c\in W_2 | c\ge t(\overset{\smile}{\mathrm{B}})\right\}.$$

Thus, because of $s(\overset{\smile}{\mathrm{B}})=s(B)+1=t(G)$ and $t(\overset{\smile}{\mathrm{B}})=t(B)=s(A)$ (Remark 5.10) we get

$$\#\{c\in W_1 | c\le t(G)\}=\#\{c\in W_2 | c\ge s(A)\}.$$

Altogether, $g(G)-(n-2k)=g(A)$ follows.

d) Since in $eC(w)$ the right endpoints of the right green arcs are at $n+1,n+2,\dots$, they are at $n+2,n+3,\dots$ in $depG(w)$, thus by c) we have

$$t(A)=n+1+g(A)=n+1+g(G)-(n-2k)=g(G)+2k+1. \tag{5.3}$$

Therefore, we get

$$\begin{aligned}
t(G)-b(G)+b(B) &= t(G)-\frac{1}{2}\big(t(G)-s(G)\big)+\frac{1}{2}\big(t(B)-s(B)\big)\\
&= t(G)-\frac{1}{2}\big(t(G)+g(G)\big)+\frac{1}{2}\big(s(A)-(t(G)-1)\big)\\
&= \frac{1}{2}\big(2t(G)-t(G)-g(G)+s(A)-t(G)+1\big)\\
&= \frac{1}{2}\big(-g(G)+s(A)+1\big)\\
&\overset{(5.3)}{=} \frac{1}{2}\big(2k-t(A)+s(A)+2\big)\\
&= k-\frac{1}{2}\big(s(A)-t(A)\big)+1\\
&= k+1-b(A).
\end{aligned}$$

e) Since $t(C)\in W_2$ is maximal with $t(C)<t(B)$, by Lemma 5.23 a) we conclude that $s(C)+1\in W_1$ is minimal with $s(C)+1>s(B)+1$. Therefore, $t(G')=s(C)+1$ is minimal with $t(G')>t(G)=s(B)+1$. Thus, by Lemma 5.23 c), we get $s(G')=s(G)-1$. Hence, we have

$$\begin{aligned}
b(G') &= \frac{1}{2}\big(t(G')-s(G')\big)=\frac{1}{2}\big(s(C)+1-(s(G)-1)\big)\\
&= \frac{1}{2}\big(s(C)-s(G)+2\big).
\end{aligned}$$

So we get

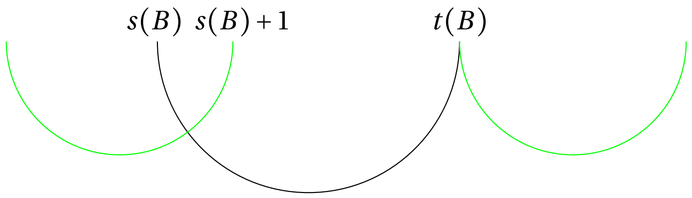
$$\begin{aligned}
&b(G')+b(C)-1+\frac{1}{2}\big(t(B)-1-t(C)\big)\\
&\quad= \frac{1}{2}\big(2b(G')+2b(C)-2+t(B)-1-t(C)\big)\\
&\quad= \frac{1}{2}\big(s(C)-s(G)+2+t(C)-s(C)-2+t(B)-1-t(C)\big)\\
&\quad= \frac{1}{2}\big(-s(G)-1+t(B)\big)\\
&\quad= \frac{1}{2}\big(s(B)+1-2-s(G)+t(B)-s(B)\big)\\
&\quad= \frac{1}{2}\big(t(G)-s(G)+t(B)-s(B)\big)-1\\
&\quad= b(G)-b(B)-1
\end{aligned}$$

and

$$
\begin{aligned}
t(G') - b(G') + b(C) + \frac{1}{2}\big(t(B) - 1 - t(C)\big) \\
&= \frac{1}{2}\big(2t(G') - 2b(G') + 2b(C) + t(B) - t(C) - 1\big) \\
&= \frac{1}{2}\Big(2\big(s(C)+1\big) - \big(s(C) - s(G) + 2\big) + \big(t(C) - s(C)\big) + t(B) - t(C) - 1\Big) \\
&= \frac{1}{2}\big(s(G) + t(B) - 1\big) \\
&= \frac{1}{2}\big(s(B) + 1 + s(G) + t(B) - s(B) - 2\big) \\
&= \frac{1}{2}\big(t(G) + s(G) + 2b(B) - 2\big) \\
&= \frac{1}{2}\big(2t(G) - t(G) + s(G)\big) + b(B) - 1 \\
&= t(G) - b(G) + b(B) - 1.
\end{aligned}
$$

□

Finally, we come to the proof of Theorem 5.20. For this, one should keep in mind the following picture, which shows the situation the arcs are in:



Here the black arc B is the one in $depG(S(w))$ with $s(B)+1 \in W_1$ and $t(B) \in W_2$. The green arcs are in $depG(w)$. The arcs arise as described in the proof of Lemma 5.23.

Proof of Theorem 5.20. Again, let $W_1 = (w_\wedge \cap S(w)_\vee) \smallsetminus w_\times$, $W_2 = (w_\vee \cap S(w)_\wedge) \smallsetminus w_\times$.

By Theorem 4.6 and Theorem 5.8 we only have to show, that for N-invariant flags the conditions of $depG(w)$ are equivalent to the conditions of $depG(S(w))$ together with the additional condition $F_i = F_i(w)$ for $i \in W_1$.

By Lemma 5.23 e), f) we only have to check this for green arcs G in $depG(w)$ with $t(G) \in W_1$ or $s(G) \in W_2$, respectively.

Let G be a green arc in $depG(w)$ with $t(G) \in W_1$ minimal. Since by Lemma 5.23 e), f) the arcs nested inside G coincide in $depG(w)$ and $depG(S(w))$, the following holds in both graphs

$$
\begin{aligned}
F_{t(G)-1} &\overset{(1)}{=} N^{\frac{1}{2}\left(t(G)-g(G)\right)}\left(\operatorname{im} N^{n-k-g(G)+1}\right) \\
&= N^{-\left(b(G)-g(G)\right)}\left\langle e_1, \ldots, e_{g(G)-1}\right\rangle \\
&= \left\langle e_1, \ldots, e_{b(G)-1}, f_1, \ldots, f_{b(G)-g(G)}\right\rangle.
\end{aligned}
$$

Here (1) holds because of Remark 5.21, Lemma 5.11 e) and $g(G)-1 \leq 2n-k$ by the same considerations as in the Proof of Theorem 5.8 since all arcs nested inside G are also in $depG(S(w))$.

The additional condition for $depG(S(w))$ means by definition of $\mathcal{F}_\bullet(w)$ and Lemma 5.25 a) that

$$\begin{aligned} F_{t(G)} = F_{t(G)}(w) &= \langle e_1, \ldots e_{b(G)}, f_1, \ldots, f_{b(G)-g(G)} \rangle \\ &= F_{t(G)-1} + \langle e_{b(G)} \rangle, \end{aligned}$$

which agrees with the condition for $depG(w)$, since $F_{t(G)-1}$ is the same in both conditions.

Now consider G with $t(G) \in W_1$ not minimal. By Lemma 5.23 c) we get, that for M with

$$t(M) = \max\{ t(B) | B \neq G \text{ is a green arc nested inside } G\}$$

also $t(M) \in W_1$ holds. Thus, by Remark 5.21 and Proposition 5.13 there is a black arc sequence from $t(M)$ to $t(G)-1$. By induction we have

$$\begin{aligned} F_{t(G)-1} &= N^{-\frac{1}{2}\left(t(G)-1-t(M)\right)} \langle e_1, \ldots, e_{b(M)}, f_1, \ldots, f_{b(M)-g(M)} \rangle \\ &= N^{-\left(b(G)-1-b(M)\right)} \langle e_1, \ldots, e_{b(M)}, f_1, \ldots, f_{b(M)-g(M)} \rangle \\ &= \langle e_1, \ldots, e_{b(G)-1}, f_1, \ldots, f_{t(G)-b(G)} \rangle \end{aligned}$$

in $depG(S(w))$ together with the additional condition and also in $depG(w)$, since by Lemma 5.25 b) we have $\frac{1}{2}\left(t(G)-1-t(M)\right) = b(G)-1-b(M)$ and $b(M)-g(M)+b(G)-1-b(M) = t(G)-b(G)$. So, again by the definition of $\mathcal{F}_\bullet(w)$ and Lemma 5.25 a), we get

$$F_{t(G)} = F_{t(G)}(w) = \langle e_1, \ldots e_{b(G)}, f_1, \ldots, f_{b(G)-g(G)} \rangle = F_{t(G)-1} + \langle f_{b(G)} \rangle,$$

and again both conditions agree.

Now we consider the other difference between $depG(w)$ and $depG(S(w))$:
Let B be a black arc in $depG(S(w))$ with $t(B) \in W_2$ and $b(B)$ minimal. By Lemma 5.23 d) in $depG(w)$ there are green arcs G and A with $t(G) = s(B)+1$ and $s(A) = t(B)$.

By Proposition 5.13 d) there is a black arc sequence from $s(B)+1$ to $t(B)-1$ in $depG(S(w))$, which is also in $depG(w)$ because of the minimality and by Lemma 5.23 e). By the above considerations we have in $depG(S(w))$ together with the additional condition as well as in $depG(w)$ that

$$\begin{aligned} F_{s(B)+1} &= F_{t(G)} \\ &= \langle e_1, \ldots, e_{b(G)}, f_1, \ldots, f_{t(G)-b(G)} \rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} F_{t(B)-1} &= N^{-b(B)+1}(F_{s(B)+1}) \\ &= N^{-b(B)+1}\left(\langle e_1, \ldots, e_{b(G)}, f_1, \ldots, f_{t(G)-b(G)} \rangle\right) \\ &= \langle e_1, \ldots, e_{b(G)+b(B)-1}, f_1, \ldots, f_{t(G)-b(G)+b(B)-1} \rangle \end{aligned}$$

holds in $depG(S(w))$ together with the additional conditions as well as in $depG(w)$.

By choice of B in $depG(S(w))$ together with the additional condition we have

$$\begin{aligned} F_{t(B)} &= N^{-b(B)} F_{s(B)} = N^{-b(B)}\left(\left\langle e_1,\dots,e_{b(G)-1}, f_1,\dots,f_{t(G)-b(G)}\right\rangle\right) \\ &= \left\langle e_1,\dots,e_{b(G)+b(B)-1}, f_1,\dots,f_{t(G)-b(G)+b(B)}\right\rangle \\ &= F_{t(B)-1} + \left\langle f_{t(G)-b(G)+b(B)}\right\rangle. \end{aligned}$$

By Lemma 5.25 d) this is equivalent to

$$\begin{aligned} F_{s(A)} = F_{t(B)} &= F_{t(B)-1} + \left\langle f_{t(G)-b(G)+b(B)}\right\rangle \\ &= F_{s(A)-1} + \left\langle f_{k+1-b(A)}\right\rangle, \end{aligned}$$

which is the conditions for $F_{s(A)}$ that $depG(w)$ provides. Since $F_{s(A)-1} = F_{t(B)-1}$ agree in both, also the conditions for $F_{s(A)}$ agree.

Now assume B is chosen as above, but with $b(B)$ not minimal. Let C be another arc with the above properties and with the next smallest $b(C)$. By Proposition 5.13 d) there is a black arc sequence from $s(B)+1$ to $t(B)-1$ in $depG(S(w))$, which has to contain C, since arcs do not intersect. Hence, by choice of C there is a black arc sequence from $t(C)$ to $t(B)-1$ in $depG(S(w))$ as well as in $depG(w)$. Thus, by induction in both $depG(w)$ and in $depG(S(w))$ together with the additional condition we have

$$\begin{aligned} F_{t(B)-1} &= N^{-\frac{1}{2}\left(t(C)-t(B)+1\right)} F_{t(C)} \\ &= N^{-\frac{1}{2}\left(t(B)-1-t(C)\right)}\left(\left\langle e_1,\dots,e_{b(G')+b(C)-1}, f_1,\dots,f_{t(G')-b(G')+b(C)}\right\rangle\right), \end{aligned}$$

where G' is the green arc in $depG(w)$ with $t(G') = s(C)+1$. By Lemma 5.25 e) we get

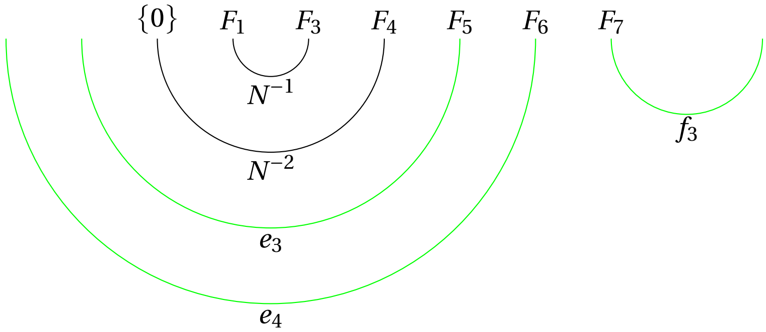
$$F_{t(B)-1} = \left\langle e_1,\dots,e_{b(G)+b(B)-1}, f_1,\dots,f_{t(G)-b(G)+b(B)-1}\right\rangle.$$

The rest works analogously to the case where $b(B)$ is minimal. □

Remark 5.26. At this point, as promised in Remark 4.9, we revisit the correlation between the dependence graph and the map of Theorem 4.8:

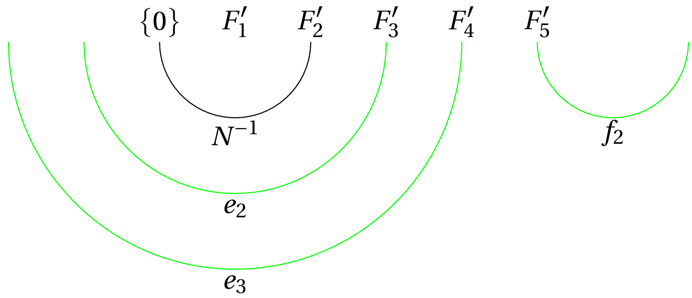
For w a row strict tableau of type $(1,\dots i-1, i+1,\dots n)$ the map $\pi : \widetilde{\mathcal{Y}}_w \to \mathcal{Y}_{p(w)}$ looks as follows on the level of dependence graphs: The arc B with $\overset{\smile}{\mathrm{B}} = \times\times$ vanishes and the other indices are adjusted accordingly.

$$depG\left(\begin{array}{|c|c|c|c|} \hline 6 & 5 & 4 & 3 \\ \hline 7 & 3 & 1 & \\ \cline{1-3} \end{array}\right) =$$



$$p\left(\begin{array}{|c|c|c|c|}\hline 6&5&4&3\\ \hline 7&3&1& \\ \hline\end{array}\right)=\begin{array}{|c|c|c|}\hline 4&3&2\\ \hline 5&1& \\ \hline\end{array}$$

$$depG\left(\begin{array}{|c|c|c|}\hline 4&3&2\\ \hline 5&1& \\ \hline\end{array}\right)=$$



Using the above discussion, we now connect a generalised irreducible component to an irreducible component associated to a cup diagram that arises from the one of the generalised component by deleting all green cups. We will need this reduction again in Section 7.

Definition 5.27. For C an extended cup diagram we denote by $depG(C)$ the dependence graph that arises from it by Definition 5.17.

Definition 5.28. Let C be an extended cup diagram. We define a *black block* to be an interval $[s,s+2t-1]\subset\{1,\dots,n\}$ only consisting of endpoints of black cups and containing both endpoints for each such cup. We denote the corresponding black cups (with adjusted numbering) by $C|_{[s,s+2t-1]}$.

Lemma 5.29. *Let C be an extended cup diagram without $\times\times$.*
Let $(F_1\subset\cdots\subset F_n)$ be a N-invariant flag satisfying the conditions of $depG(C)$. Let N' : $F_{s+2t-1}/F_{s-1}\to F_{s+2t-1}/F_{s-1}$ be the induced map. Then for each black block $[s,s+2t-1]$ the flag $\left(F_s/F_{s-1}\subset\cdots\subset F_{s+2t-1}/F_{s-1}\right)$ is an N'-invariant flag satisfying the conditions of $depG(C|_{[s,s+2t-1]})$.
Conversely, assume we have a flag $(F_1\subset\cdots\subset F_n)$ with $N: F_n\to F_n$. Let there be a covering of the black cups in C by disjoint black blocks $[s,s+2t-1]$ such that $\left(F_s/F_{s-1}\subset\cdots\subset F_{s+2t-1}/F_{s-1}\right)$ is a N'-invariant flag satisfying the conditions of $depG(C|_{[s,s+2t-1]})$, where N' is induced by N. Assume further that $F_{t(G)}=F_{t(G)-1}+\left\langle e_{b(G)}\right\rangle$ holds for all left green cups G and $F_s(H)=F_{s(H)-1}+\left\langle f_{k+1-b(H)}\right\rangle$ holds for all right green cups H. Then $(F_1\subset\cdots\subset F_n)$ is a N-invariant flag satisfying the conditions of $depG(C)$.

Proof. As above, from the conditions of $depG(C)$ we get $F_{s-1}=\langle e_1,\dots,e_r,f_1,\dots,f_q\rangle$ and $F_{s+2t-1}=\langle e_1,\dots,e_{r+t},f_1,\dots,f_{q+t}\rangle$. We have $N'^{-1}(F_j+F_{s-1})=N^{-1}(F_j)+F_{s-1}$ for $s-1\le j\le s+2t-3$ and $N'(F_j+F_{s-1})=N(F_j)+F_{s-1}$ for $s\le j\le s+2t-1$. Hence, the first assertion follows.

For the second part, we use that the conditions of $depG(C)$ are satisfied by definition or with the same computations as for the first part. By the conditions we inductively get $F_{s+2t-1}=\langle e_1,\dots,e_{r'},f_1,\dots,f_{q'}\rangle$, since the first F_{s-1} is either $=0$ or determined by green arcs. In particular, as computed in the main theorem of this section, if $s+2t-1=t(G)-1$ for G a left green cup, then $e_{r'}=e_{b(G)-1}$, and if $s+2t-1=s(H)-1$ for H a right green cup, then $f_{q'} = f_{k-b(H)}$. For a F_j determined by a green cup, we have either $F_j = F_{j-1} + \langle e_{b(G)} \rangle$ or $F_j = F_{j-1} + \langle f_{k+1-b(G)} \rangle$. Therefore, we have $NF_j \subset F_{j-1}$ for the F_j's determined by green cups. For the remaining F_j's the N-invariance holds because of the computation from the first part; in particular we have $NF_s \subset F_{s-1}$ since $N'(F_s + F_{s-1}) \subset F_{s-1} + F_{s-1}$. □

Proposition 5.30. *Let w be row strict tableau of type $(1,\dots,n)$ of shape $(n-k,k)$ and let n' be twice the number of black cups in $eC(w)$. Then there is a standard tableau S of type $(1,\dots,n')$ and shape $(\frac{n'}{2},\frac{n'}{2})$ such that we have a bijective morphism of varieties $\mathcal{Y}_w \to Y_S$.*

Proof. Let S be the standard tableau corresponding to the cup diagram that arises from $eC(w)$ by deleting the green cups. In $depG(w)$ let

$$\begin{aligned} J_1 &:= \{ j \in \{1,\dots,n\} | j = t(G) \text{ for } G \text{ a green arc}\} \text{ and} \\ J_2 &:= \{ j \in \{1,\dots,n\} | j = s(G) \text{ for } G \text{ a green arc}\}. \end{aligned}$$

Let $J := J_1 \cup J_2$. We define $V_1 := \langle e_{b(G)} | t(G) = j \text{ for some } j \in J_1 \rangle$ and $V_2 := \langle f_{k+1-b(G)} | s(G) = j \text{ for some } j \in J_2 \rangle$. Furthermore, let $\alpha : \mathbb{C}^n \to \mathbb{C}^n/(V_1 \oplus V_2)$ be the projection.

Now we define

$$\begin{aligned} \varphi : \mathcal{Y}_w &\to Y_S \\ (F_1 \subset \cdots \subset F_n) &\mapsto \left(\alpha(F_{j_1}) \subset \cdots \subset \alpha(F_{j_r})\right) \end{aligned}$$

where $\{j_1,\dots,j_r\} = \{1,\dots,n\} \setminus J$.

Then we have $\alpha(F_{j_r}) = \alpha(F_n) = \mathbb{C}^n/(V_1 \oplus V_2) = \mathbb{C}^{n'}$.

Let $N' : \mathbb{C}^{n'} \to \mathbb{C}^{n'}$ be defined by $N' e'_i = e'_{i-1}$ and $N' f'_j = f'_{j-1}$ for $i, j = 1, \dots, \frac{n'}{2}$, where

$$\{e'_1,\dots,e'_{n'}\} = \{e_1 + (V_1 \oplus V_2), \dots, e_{n-k} + (V_1 \oplus V_2)\} \setminus \{e_j + (V_1 \oplus V_2) | e_j \in V_1\}$$

and the same for the f'_j's.

Then $\left(\alpha(F_{j_1}) \subset \cdots \subset \alpha(F_{j_r})\right)$ is N'-invariant and satisfies the conditions of $depG(S)$, since the conditions of Lemma 5.29 hold for the black blocks which are segregated by green cups. The corresponding quotients are equal and the restrictions of N or N', respectively, are acting in the same way.

Furthermore, the map given above is a morphism of varieties by an argumentation analogous to the one in the proof of Theorem 4.8.

Analogously, using Lemma 5.29 we can construct an inverse map by defining the subspaces associated to a green cup as given by $depG(w)$.

□

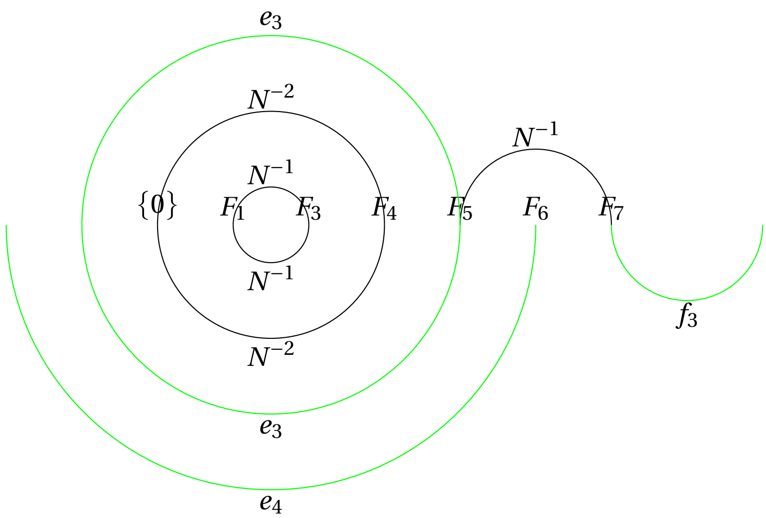
5.3 Dependence graphs for intersections

In this section we apply the result of the last chapter to get a description of the structure of intersections of generalised irreducible components via associated dependence graphs.

Definition 5.31. Let w and w' be row-strict tableaux of type $(i_1, \ldots, i_m)$. The dependence graph for (w, w'), *depG*(w, w'), is constructed by reflecting the dependence graph for w' across the horizontal axis and putting it on top of the dependence graph for w.

Example 5.32. Let again $w = \begin{array}{|c|c|c|c|}\hline 6 & 5 & 4 & 3 \\ \hline 7 & 3 & 1 \\ \cline{1-3}\end{array}$ and let $w' = \begin{array}{|c|c|c|c|}\hline 7 & 5 & 4 & 3 \\ \hline 6 & 3 & 1 \\ \cline{1-3}\end{array}$ (which is the S from Example 5.6).

depG(w, w') =



Definition 5.33. Let w, w' be row strict tableaux of type $(i_1, \ldots, i_m)$. A flag $(F_{i_1} \subset \cdots \subset F_{i_m})$ *satisfies the conditions of depG*(w, w') if we have

1. if the node labelled F_i $(i > 0)$ is connected to a node labelled F_j with $i < j$ via a black arc labelled N^{-l}, then $F_j = N^{-l} F_i$
2. if the node labelled F_i is the endpoint of a green arc labelled e_l, then $F_i = F_{i-1} + \langle e_l \rangle$
3. if the node labelled F_i is the starting point of a green arc labelled f_l, then $F_i = F_{i-1} + \langle f_l \rangle$

Corollary 5.34. *Let w and w' be row-strict tableaux of type $(i_1, \ldots, i_m)$.*

Then $\widetilde{\mathcal{Y}}_w \cap \widetilde{\mathcal{Y}}_{w'}$ consists of all N-invariant flags satisfying the conditions of the dependence graph for (w, w').

The space F_j, $j > 0$, is independent in $\widetilde{Y}_w \cap \widetilde{Y}_{w'}$ if and only if the node labelled F_j is the node at the left end in a connected component of the dependence graph for (w, w').

Proof. $F_\bullet$ is in $\widetilde{Y}_w \cap \widetilde{Y}_{w'}$ if and only if the conditions of $\widetilde{Y}_w$ and the conditions of $\widetilde{Y}_{w'}$ are satisfied. But these conditions are given by the associated dependence graph. If the dependence graphs are put on top of each other, then both conditions are satisfied simultaneously. □

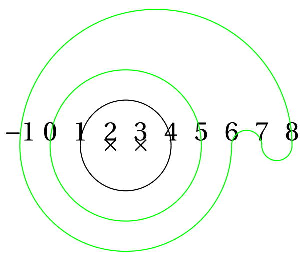
6 Circle diagrams

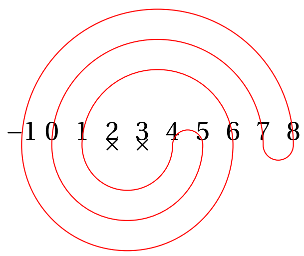
In this section we use circle diagrams to give an equivalent condition for the fact that the intersection of two generalised irreducible components is empty.

Following [Str09, 5.4] we construct circle diagrams out of cup diagrams and colour them:

Definition 6.1. Let w, w' be row strict tableaux. We define $CC(w, w')$, the circle diagram for (w, w'), as follows: We reflect $eC(w')$ and put it on top of $eC(w)$. If there are more points in $eC(w)$ than in $eC(w')$ or vice versa, we connect in $eC(w')$ the ones on the right with the ones on the left via an green arc in the only possible crossingless way. The construction up to here is called $eC(w, w')$. If there is at least one green arc in a connected component, we colour the whole component green. If there is more than one left outer point, i.e. a point p with $p < 1$, or more than one right outer point, i.e. a point p with $p > n$, in a connected component, we colour the whole component red.

Example 6.2. $w = \begin{array}{|c|c|c|c|}\hline 6 & 5 & 4 & 3 \\ \hline 7 & 3 & 1 & \\ \cline{1-3}\end{array}$, $w' = \begin{array}{|c|c|c|c|}\hline 7 & 5 & 4 & 3 \\ \hline 6 & 3 & 1 & \\ \cline{1-3}\end{array}$, $w'' = \begin{array}{|c|c|c|c|}\hline 7 & 6 & 5 & 3 \\ \hline 4 & 3 & 1 & \\ \cline{1-3}\end{array}$

$CC(w, w') =$ 

$CC(w, w'') =$ 

Remark 6.3. The nodes at the left end in a black connected component of the dependence graph for (w, w') coincide with the left points of the black circles in $CC(w, w')$. This follows from the same argument as in Remark 5.12 and Remark 5.22: k is left point of a black circle if and only if there is no cup ending at k in $eC(w)$ and in $eC(w')$ and k is not a double entry, which is equivalent to $k \in w_\vee \cap w'_\vee \setminus w_\times$, i.e. $k \notin w_\wedge \cup w'_\wedge$. This holds if and only if there is no arc B with $k = t(B)$ in *depG*(w) and *depG*(w'), which means that k is the left end of a connected component in *depG*(w, w').

Therefore, by Corollary 5.34 the number of independents in $\widetilde{\mathcal{Y}}_w \cap \widetilde{\mathcal{Y}}_{w'}$ is the same as the number of black circles in $CC(w, w')$.

In the following section the theorems often have the assumption $\widetilde{\mathcal{Y}}_w \cap \widetilde{\mathcal{Y}}_{w'} \neq \varnothing$. The following theorem gives an equivalent condition for this.

Theorem 6.4. *Let w, w' be row strict tableaux of type $(i_1, \dots, i_m)$. Then $\widetilde{\mathcal{Y}}_w \cap \widetilde{\mathcal{Y}}_{w'} = \varnothing$ if and only if there is at least one red circle in $CC(w, w')$.*

Before we can prove the theorem we need some additional notation and lemmata.

Definition 6.5. For B an arc in *depG*(w, w') we still denote by $s(B)$ the left endpoint and by $t(B)$ the right endpoint of B, independent of the fact whether B is a cap or a cup.

An *extended arc sequence* in *depG*(w, w') is a sequence of arcs $B_1, \dots, B_l$ in *depG*(w, w') such that for $1 \leq i \leq l-1$ exactly one of the following conditions hold:

$$t(B_i) = s(B_{i+1}), \quad t(B_i) = t(B_{i+1}), \quad s(B_i) = s(B_{i+1}), \quad s(B_i) = t(B_{i+1}).$$

In *depG*(w, w') an *extended arc sequence from a to b* is an extended arc sequence with

$$a = \begin{cases} s(B_1) \\ t(B_1) \end{cases} \quad \text{and } b = \begin{cases} s(B_l) \\ t(B_l) \end{cases}.$$

In $eC(w, w')$ we define a extended arc sequence analogously.

For G a green arc in *depG*(w, w') let

$$r(G) = \begin{cases} t(G), & \text{if } G \text{ is a left green arc} \\ s(G), & \text{if } G \text{ is a right green arc} \end{cases} \qquad \text{and}$$

$$u(G) = \begin{cases} s(G), & \text{if } G \text{ is a left green arc} \\ t(G), & \text{if } G \text{ is a right green arc} \end{cases}$$

For B an arc in *depG*(w, w') or $eC(w, w')$, respectively, let

$$p(B) = \begin{cases} 0, & \text{if } B \text{ above the x-axis} \\ 1, & \text{if } B \text{ under the x-axis.} \end{cases}$$

Remark 6.6. Let $B_1, \dots, B_l$ be an extended arc sequence in $eC(w, w')$. Then $p(B_{i+1}) \equiv p(B_i) + 1 \mod 2$, since each point is either starting point or endpoint of a cup in $eC(w)$ and of a cup in $eC(w')$. In particular, each circle in $CC(w, w')$ consists of an even number of arcs.

Lemma 6.7. *If there is a black extended arc sequence from a to b in* depG(w, w')*, then we have $F_b = N^{-\frac{1}{2}(b-a)} F_a$ for all flags in $\widetilde{\mathcal{Y}}_w \cap \widetilde{\mathcal{Y}}_{w'}$.*

Proof. For an extended arc sequence $B_1, \dots, B_l$ from a to $b \geq a$ let

$$x_i = \begin{cases} 1, & \text{if } s(B_i) = t(B_{i-1}) \\ 1, & \text{if } t(B_i) = s(B_{i-1}) \\ -1, & \text{if } t(B_i) = t(B_{i-1}) \\ -1, & \text{if } s(B_i) = s(B_{i-1}). \end{cases}$$

for $2 \leq i \leq l$ and $x_1 = \begin{cases} 1, & \text{if } s(B_1) = a \\ -1, & \text{if } t(B_1) = a \end{cases}$. Since all intermediate summands kill each other, we have

$$\sum_i x_i b(B_i) = \frac{1}{2}(b-a).$$

Since for dimensional reasons we have that $F_{t(B_i)} = N^{-b(B_i)} F_{s(B_i)}$ is equivalent to $F_{s(B_i)} = N^{b(B_i)} F_{t(B_i)}$, we get $F_b = N^{-\frac{1}{2}(b-a)} F_a$. □

Lemma 6.8. *Let $B_1, \dots, B_l$ be a black extended arc sequence from a to b in $eC(w, w')$.*
If $p(B_1) = p(B_l)$, then there are two black extended arc sequences in $\mathrm{depG}(w, w')$*, one from $a-1$ to b and one from a to $b-1$.*
If $p(B_1) \neq p(B_l)$, then there are two black extended arc sequences in $\mathrm{depG}(w, w')$*, one from $a-1$ to $b-1$ and one from a to b.*

Proof. Let $\widehat{B}_1, \dots, \widehat{B}_l$ be the images of the B_i in *depG*(w, w').

By construction of *depG*(w, w') and by Remark 5.21 and Proposition 5.13 there is a black arc sequence from $s(\widehat{B}_i)+1 = s(B_i)$ to $t(\widehat{B}_i)-1 = t(B_i)-1$ for each arc $\widehat{B}_i$. Therefore, if we consider B_i as an arc sequence from c to d, then there is a black extended arc sequence from $c-1$ to d and one from c to $d-1$ in *depG*(w, w').

Now we consider the different cases how an extended arc sequence in $eC(w, w')$ is put together: For $i \geq 2$ we have:

If $s(B_i) = t(B_{i-1})$, then there is a black extended arc sequence from $s(B_{i-1}) = s(\widehat{B}_{i-1})+1$ to $t(\widehat{B}_{i-1}) - 1 = s(\widehat{B}_i)$ in *depG*(w, w') and one from $t(\widehat{B}_{i-1}) = s(\widehat{B}_i)+1$ to $t(\widehat{B}_i) - 1 = t(B_i) - 1$. Together with $\widehat{B}_{i-1}$ and $\widehat{B}_i$ there is a black extended arc sequence from $s(B_{i-1}) = s(\widehat{B}_{i-1})+1$ to $t(\widehat{B}_i) = t(B_i)$ and one from $s(B_{i-1})-1 = s(\widehat{B}_{i-1})$ to $t(\widehat{B}_i)-1 = t(B_i)-1$.

If $t(B_i) = s(B_{i-1})$, then analogously in *depG*(w, w') there is a black extended arc sequence from $t(\widehat{B}_{i-1}) = t(B_{i-1})$ to $s(B_i) = s(\widehat{B}_i)+1$ and one from $t(\widehat{B}_{i-1})-1 = t(B_{i-1})-1$ to $s(B_i)-1 = s(\widehat{B}_i)$.

If $s(B_i) = s(B_{i-1})$, then there is a extended black arc sequence from $t(B_{i-1}) = t(\widehat{B}_{i-1})$ to $t(B_i) = t(\widehat{B}_i)$ in *depG*(w, w') given by $\widehat{B}_i$ and $\widehat{B}_{i-1}$. Furthermore, there is a black arc sequence from $s(\widehat{B}_{i-1}) - 1 = s(B_{i-1})$ to $t(B_{i-1}) - 1 = t(\widehat{B}_{i-1}) - 1$ and one from $s(\widehat{B}_i) - 1 = s(B_i)$ to $t(B_i) - 1 = t(\widehat{B}_i) - 1$, so altogether we have an extended arc sequence from $t(B_{i-1}) - 1 = t(\widehat{B}_{i-1}) - 1$ to $t(B_i) - 1 = t(\widehat{B}_i) - 1$.

If $t(B_i) = t(B_{i-1})$, then analogously there is a black extended arc sequence from $s(B_{i-1}) - 1 = s(\widehat{B}_{i-1})$ to $s(B_i) - 1 = s(\widehat{B}_i)$ and one from $s(B_{i-1}) = s(\widehat{B}_{i-1}) + 1$ to $s(B_i) = s(\widehat{B}_i) + 1$.

Consider B_{i-1}, B_i as extended arc sequence from c to d. Then in each of the above cases we get an extended arc sequence from $c-1$ to $d-1$ and one from c to d.

Therefore, for even l we have a black extended arc sequence from $a-1$ to $b-1$ and one from a to b, and for odd l we have a black extended arc sequence from $a-1$ to b and one from a to $b-1$.

By Remark 6.6 we know that l is odd if $p(B_1) = p(B_l)$ and l is even if $p(B_1) \neq p(B_l)$, so the assertion follows. □

Proof of Theorem 6.4. Assume $CC(w, w')$ contains a red circle. Consider this circle in $eC(w, w')$, i.e. when the colours are the colours from $eC(w)$ and $eC(w')$.

We show that there are two arcs H_1, H_2 in this circle in $eC(w, w')$ with $p(H_1) = p(H_2)$ which are either both left green arcs or both right green arcs and $r(H_1)$ and $r(H_2)$ are connected by a black arc sequence.

Since there are no arcs connecting two outer left or right points, respectively, among each other, for each green arc H in $eC(w, w')$ there is a green arc H' with $p(H) \equiv p(H') + 1 \mod 2$ and $u(H) = u(H')$. Let $L_1, \ldots, L_N$ be the left green arcs in this circle with $p(L_i) = 1$ where $r(L_1) > \cdots > r(L_N)$. Let $L'_1, \ldots, L'_N$ be the associated left green arcs with $u(L_i) = u(L'_i)$ and $p(L'_i) = 0$. Analogously we define $R_1, \ldots, R_M$ and $R'_1, \ldots, R'_M$. Note that here the arcs, that do not come from the cup diagrams but are added by the construction, are counted twice.

By assumption we have $N \geq 2$ or $M \geq 2$. Since the arcs $L_1, \ldots, L_N$, $L'_1, \ldots, L'_N$, $R_1, \ldots, R_M$ and $R'_1, \ldots, R'_M$ are all part of one circle, every one of them is connected by a black extended arc sequence (possibly of length 0) to another one. If we have one of the arcs which are counted twice represented for example by L_a and R_b, we think of L_a and R_b to be connected by an arc sequence of length 0.

We consider where the arc L'_1 is connected to by a black extended arc sequence. Since the arcs are not allowed to intersect each other, one connection restricts the possibilities for the other arcs. We distinguish the following cases: L'_1 is connected to

- L'_a for $a \neq 1$: Then we have found what we were looking for.
- L_1: Then L_1, L'_1 and the arc sequence build a circle which does not contain the other green arcs, so this is a contradiction.
- L_a for $a \neq 1$: Then $L_{a+1}, \ldots, L_N$, $L'_2, \ldots, L'_N$ have to be connected among each other. Since $N-1 > N-a$, we conclude that there are L'_r and L'_s which are connected or the connection is not possible. So are either finished or get a contradiction.
- R_a: Then $R'_1, \ldots, R'_M, R_{a+1}, \ldots, R_M$ have to be connected among each other. Since $M-a < M$ it is either not possible to connect the arcs of the circle or we have found the arcs.
- R'_a for $a \neq 1$: Then $R'_1, \ldots, R'_{a-1}$ have to be connected among each other, so we are either finished or have a contradiction.

- R_1': Then the sequence goes on with R_1. Since the situation is symmetric, the same as for L_1' holds for R_1. So we are either finished or have a contradiction or R_1 is connected to L_1. But in the last case we have a circle which only contains one outer point on each side, thus we get a contradiction.

So in each possible case we get two arcs H_1, H_2 with $p(H_1) = p(H_2)$ which are either both left green arcs or both right green arcs and $r(H_1)$ and $r(H_2)$ are connected by a black arc sequence. Because of the black arc sequence we have $r(H_1), r(H_2) \in \{1, \ldots, n\}$. Therefore, H_1 and H_2 have to exist in $eC(w)$ or in $eC(w')$.

We denote the images of H_1 and H_2 in $depG(w, w')$ by G_1 and G_2 respectively. By Lemma 6.8 in $depG(w, w')$ there is a black arc sequence from $r(G_1) - 1$ to $r(G_2)$ and one from $r(G_1)$ to $r(G_2) - 1$. Thus, by Lemma 6.7 for all flags in $\widetilde{\mathcal{Y}}_w \cap \widetilde{\mathcal{Y}}_{w'}$ we get

$$\begin{aligned} F_{r(G_2)-1} &= N^{-\frac{1}{2}\left(r(G_2)-1-r(G_1)\right)} F_{r(G_1)} \\ F_{r(G_2)} &= N^{-\frac{1}{2}\left(r(G_2)-r(G_1)+1\right)} F_{r(G_1)-1} \\ &= N^{-1} N^{-\frac{1}{2}\left(r(G_2)-1-r(G_1)\right)} F_{r(G_1)-1}. \end{aligned} \tag{6.1}$$

Since they are green arcs in $depG(w)$ or in $depG(w')$, respectively, for G_1, G_2 we have either

$$\begin{aligned} F_{t(G_1)} &= F_{t(G_1)-1} + \left\langle e_{b(G_1)} \right\rangle \\ F_{t(G_2)} &= F_{t(G_2)-1} + \left\langle e_{b(G_2)} \right\rangle \end{aligned} \tag{6.2}$$

or

$$\begin{aligned} F_{s(G_1)} &= F_{s(G_1)-1} + \left\langle f_{k+1-b(G_1)} \right\rangle \\ F_{s(G_2)} &= F_{s(G_2)-1} + \left\langle f_{k+1-b(G_2)} \right\rangle. \end{aligned} \tag{6.3}$$

From (6.1) and (6.2) we get

$$\begin{aligned} F_{t(G_2)} &= N^{-1} N^{-\frac{1}{2}\left(t(G_2)-1-t(G_1)\right)} F_{t(G_1)-1} \\ F_{t(G_2)} &= N^{-\frac{1}{2}\left(t(G_2)-1-t(G_1)\right)} \left(F_{t(G_1)-1} + \left\langle e_{b(G_1)} \right\rangle\right) + \left\langle e_{b(G_2)} \right\rangle, \end{aligned}$$

and from (6.1) and (6.3) we get

$$\begin{aligned} F_{s(G_2)} &= N^{-1} N^{-\frac{1}{2}\left(s(G_2)-1-s(G_1)\right)} F_{s(G_1)-1} \\ F_{s(G_2)} &= N^{-\frac{1}{2}\left(s(G_2)-1-s(G_1)\right)} \left(F_{s(G_1)-1} + \left\langle f_{k+1-b(G_1)} \right\rangle\right) + \left\langle f_{k+1-b(G_2)} \right\rangle. \end{aligned}$$

But in both cases the two equations contradict each other since in the second equation there are more e_i's or more f_i's, respectively, than in the first. Since the equations have to hold for all flags in $\widetilde{\mathcal{Y}}_w \cap \widetilde{\mathcal{Y}}_{w'}$, the intersection is empty.

Conversely, we assume that $CC(w, w')$ contains only black and green circles. Analogously to [SW08, Lemma 19] we construct a row strict tableau w'', such that $\mathcal{F}_\bullet(w'') \in \widetilde{\mathcal{Y}}_w \cap \widetilde{\mathcal{Y}}_{w'}$:

In $CC(w,w')$ we mark all points with $j \le 0$ by $\vee$ and all points with $j > n$ by $\wedge$. Now we assign a $\wedge$ or a $\vee$ to all $j \in \{1,\dots,n\} \setminus w_\times$ as follows:

If we have a black circle, we mark one point arbitrarily and then we follow the circle and alternatingly mark the points where we meet the x-axis by $\wedge$ or $\vee$, such that each arc has a $\wedge$ at one end and a $\vee$ at the other.

This is possible, since by Remark 6.6 each circle has a even number of arcs.

For green circles we start with a point already marked and go on as for black circles.

For green circles with only one predetermined marking this is possible for the same reason as for black circles. Now we consider a green circle with two outer points, one left and one right, which are already marked. We distinguish two cases: In the first case an additional arc from an outer left point to an outer right point was added by construction of $CC(w,w')$. In this case the markings of the two outer points are compatible. In the other case, two green arcs G_1, G_2 with $0 = p(G_1) \neq p(G_2) = 1$ start at the left outer point and two green arcs G_3, G_4 with $0 = p(G_3) \neq p(G_4) = 1$ end at the right outer point. Since $G_1,\dots,G_4$ are lying on a circle and the arcs do not intersect, there has to be a black extended arc sequence from G_1 to G_3 and from G_2 to G_4 in $eC(w,w')$, because there are no further green arcs on the circle. As in the proof of Lemma 6.8 we get that the two black extended arc sequences have an odd length, thus the predetermined markings of the two outer points are compatible.

Now we define a row strict tableau w'' by the fact that $j \in w''_\wedge$ if and only if j is a double entry or marked by $\wedge$ and $j \in w''_\vee$ if and only if j is a double entry or marked by $\vee$.

In Lemma 4.2 we already showed that $\mathcal{F}_\bullet(w'')$ is N-invariant. Furthermore, $\mathcal{F}_\bullet(w'')$ satisfies the conditions of $depG(w,w')$:

Let B be a black arc in $depG(w,w')$. We have

$$F_{s(B)}(w'') = \left\langle \{e_j, f_r \mid j \le t_{s(B)}, r \le b_{s(B)}\}\right\rangle,$$

where t_s is the number of indices smaller than or equal to s in the top row and similarly for b_s in the bottom row as before. Therefore, $F_{s(B)}(w'') = \langle e_1,\dots,e_a, f_1,\dots,f_b\rangle$ for suitable a,b.

In $CC(w,w')$ there are arcs or $\times\times$ nested inside $\widehat{B}$ such that no point j with $s(B)+1 = s(\widehat{B}) \le j \le t(\widehat{B}) = t(B)$ is vacant. Since each arc is marked at one end with $\wedge$ and at the other one with $\vee$ and the double entries appear in both lines of w'', we have

$$\#\{j \in w''_\wedge \mid s(B) < j \le t(B)\} = \#\{j \in w''_\vee \mid s(B) < j \le t(B)\} = b(B).$$

Therefore, we obtain

$$\begin{aligned} F_{t(B)}(w'') &= \left\langle \{e_j, f_r \mid j \le t_{t(B)}, r \le b_{t(B)}\}\right\rangle \\ &= \left\langle e_1,\dots,e_{a+b(B)}, f_1,\dots,f_{b+b(B)}\right\rangle = N^{-b(B)} F_{s(B)}. \end{aligned}$$

Now we consider a left green arc G in $depG(w,w')$. Since $s(\widehat{G})$ is marked with $\vee$, we have $t(G) = t(\widehat{G}) \in w''_\wedge$. So we get

$$\#\{j \in w''_\wedge \mid j \le t(G)\} = b(G),$$

because the left hand side is counting the number of arcs nested inside G, which by Remark 5.21 and Proposition 5.13 coincides with the left hand side. Altogether, we obtain $F_{t(G)}(w'') = F_{t(G)-1}(w'') + \langle e_{b(G)} \rangle$.

Let H be a right green arc in $depG(w, w')$. Since $t(\widehat{G})$ is marked with $\wedge$, we have $s(G) = s(\widehat{G}) \in w''_{\vee}$. Analogously to the above we obtain $\#\{j \in w''_{\vee} | j \geq s(H)\} = b(H)$. (Here we cannot use Proposition 5.13, but the proof of it: For every j in the right set there is a mate in $w''_{\wedge}$, so half of the possible places are marked by $\vee$.) Therefore

$$\#\{j \in w''_{\vee} | j \leq s(H)\} = k + 1 - b(H)$$

follows, since there are k entries in the bottom row of w''. Altogether, we get $F_{s(H)}(w'') = F_{s(H)-1}(w'') + \langle f_{k+1-b(H)} \rangle$.

Thus, the conditions of the dependence graph for (w, w') are satisfied. □

Remark 6.9. In the proof one can see a fact similar to [SW08, Lemma 19]: If there are no red circles in $CC(w, w')$, the number of fixed points contained in $\widetilde{\mathcal{Y}}_w \cap \widetilde{\mathcal{Y}}_{w'}$ is at least 2^x for x the number of black circles in $CC(w, w')$. Indeed, there are two possible choices for every black circle and only one for every green circle.

7 Iterated fibre bundles

In this section we use cup diagram decompositions to show that the generalised irreducible components and the intersection of these are iterated $\mathbb{CP}^1$-bundles.

By [GR65] to each complex projective variety X we can associated a topological Hausdorff space X^{an}, the associated analytic space with the same underlying set. If the projective variety is smooth, X^{an} is a complex manifold.

Following [Fun03] we consider iterated fibre bundles:

Definition 7.1. A space X_1 is an *iterated fibre bundle* of base type $(B_1, \ldots, B_l)$ if there exist spaces $X_1, B_1, X_2, B_2, \ldots, X_l, B_l, X_{l+1} = pt$ and maps $p_1, p_2, \ldots, p_l$ such that $p_j : X_j \to B_j$ is a fibre bundle with typical fibre X_{j+1}. Here fibre bundle means topological fibre bundle in the sense of [Hat03].

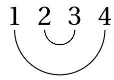
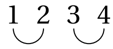
The following theorem is the main theorem of this section. It generalises [SW08] and [Fun03].

Theorem 7.2. *Let w, w' be row strict tableaux of type $(i_1, \ldots, i_m)$ and assume $\widetilde{\mathcal{Y}}_w \cap \widetilde{\mathcal{Y}}_{w'} \neq \emptyset$. Then $(\widetilde{\mathcal{Y}}_w)^{an}$ and $(\widetilde{\mathcal{Y}}_w \cap \widetilde{\mathcal{Y}}_{w'})^{an}$ are iterated bundles of base type $(\mathbb{CP}^1, \ldots, \mathbb{CP}^1)$, where there are as many terms as there are independent nodes in the associated dependence graph.*

For the rest of the section we drop the ^{an} from the notation and always work with the analytic spaces.

For the proof of Theorem 7.2 we first take a look at two important special cases:

Example 7.3. Let $n = 4$, $k = 2$. As mentioned in Example 5.3, in this case there are two standard tableaux of type $(1,2,3,4)$: $S_1 = \begin{array}{|c|c|}\hline 4 & 3 \\ \hline 2 & 1 \\ \hline \end{array}$ and $S_2 = \begin{array}{|c|c|}\hline 4 & 2 \\ \hline 3 & 1 \\ \hline \end{array}$.
For these we have the following associated extended cup diagrams:

$eC(S_1) =$  $eC(S_2) =$ 

1) Firstly, we consider Y_{S_2}: As one can see for example from the associated dependence graph, we have

$$Y_{S_2} = \{F_1 \subset N^{-1}(F_0) \subset F_3 \subset N^{-2}(F_0)\}.$$

We define $p : Y_{S_2} \to \mathbb{P}(N^{-1}(F_0)) = \mathbb{CP}^1$ via

$$\big(F_1 \subset N^{-1}(F_0) \subset F_3 \subset N^{-2}(F_0)\big) \mapsto F_1.$$

This is a trivial fibre bundle with fibre a component of a smaller Springer fibre, namely the one with associated cup diagram  :

Let $N_1 := N|_{\langle e_1, f_1\rangle}$ and let N_2 be the map induced by N on $\mathbb{C}^4/\ker N \to \mathbb{C}^4/\ker N$. We take as trivialising neighbourhood the whole base space and get the following trivialisation:

$$\begin{aligned}
\{F_1 \subset N_1^{-1}(\{0\})\} \times \{G_1 \subset N_2^{-1}(\{0\})\} &\to p^{-1}\big(\{F_1 \subset N^{-1}(\{0\})\}\big) \\
\big(F_1 \subset N^{-1}(\{0\}), G_1 \subset N_2^{-1}(\{0\})\big) &\mapsto \big(F_1 \subset N_1^{-1}(\{0\}) \subset N_1^{-1}(\{0\}) + G_1 \\
&\qquad \subset N_1^{-1}(\{0\}) + N_2^{-1}(\{0\}) = \mathbb{C}^2 \oplus \mathbb{C}^2 = \mathbb{C}^4\big) \\
\big(F_1 \subset N_1^{-1}(\{0\}), F_3/\ker N \subset \mathbb{C}^4/\ker N\big) &\mapsfrom \big(F_1 \subset N^{-1}(\{0\}) \subset F_3 \subset \mathbb{C}^4\big)
\end{aligned}$$

This is a homeomorphism and commutes with the projections.

2) Now we consider the space $Y_{S_1} = \{F_1 \subset F_2 \subset N^{-1}(F_1) \subset N^{-2}(\{0\})\}$. Again we define $p : Y_{S_1} \to \mathbb{P}\big(N^{-1}(F_0)\big) = \mathbb{CP}^1$ via

$$\big(F_1 \subset F_2 \subset N^{-1}(F_1) \subset N^{-2}(\{0\})\big) \mapsto F_1.$$

We choose the standard covering of $\mathbb{CP}^1$:

$$U_1 := \{(x : y) | x \neq 0\} \text{ and } U_2 := \{(x : y) | y \neq 0\}$$

We consider $(1 : \lambda) \in U_1$. With our identifications this corresponds to $\langle e_1 + \lambda f_1\rangle \subset \langle e_1, f_1\rangle$. We have $N^{-1}\langle e_1 + \lambda f_1\rangle = \langle e_1, f_1, e_2 + \lambda f_2\rangle = \langle e_1 + \lambda f_1, f_1, e_2 + \lambda f_2\rangle$, therefore $N^{-1}\langle e_1 + \lambda f_1\rangle / \langle e_1 + \lambda f_1\rangle = \langle f_1, e_2 + \lambda f_2\rangle$. We denote by α_λ the isomorphism $\mathbb{C}^2 \to \langle f_1, e_2 + \lambda f_2\rangle$ given by mapping the standard basis to $f_1, e_2 + \lambda f_2$. We get the following trivialisation for U_1:

$$\begin{aligned}
&U_1 \times \{L \subset \mathbb{C}^2\} \to p^{-1}(U_1) \\
&\big((1 : \lambda), L \subset \mathbb{C}^2\big) \mapsto \big(\langle e_1 + \lambda f_1\rangle \subset \langle e_1 + \lambda f_1\rangle + \alpha_\lambda(L) \subset N^{-1}(\langle e_1 + \lambda f_1\rangle) \subset \mathbb{C}^4\big)
\end{aligned}$$

This map commutes with the projections, and it is a homeomorphism with inverse

$$\begin{aligned}
p^{-1}(U_1) &\to U_1 \times \{L \subset \mathbb{C}^2\} \\
&\Big(\langle e_1 + \lambda f_1\rangle \subset F_2 \subset N^{-1}(\langle e_1 + \lambda f_1\rangle) \subset \mathbb{C}^4\Big) \\
&\quad \mapsto \big(F_1 \subset N^{-1}(F_0)\big) \\
&\qquad \times \Big(\alpha_\lambda^{-1}(F_2/\langle e_1 + \lambda f_1\rangle) \subset \alpha_\lambda^{-1}(N^{-1}(\langle e_1 + \lambda f_1\rangle)/\langle e_1 + \lambda f_1\rangle) = \mathbb{C}^2\Big).
\end{aligned}$$

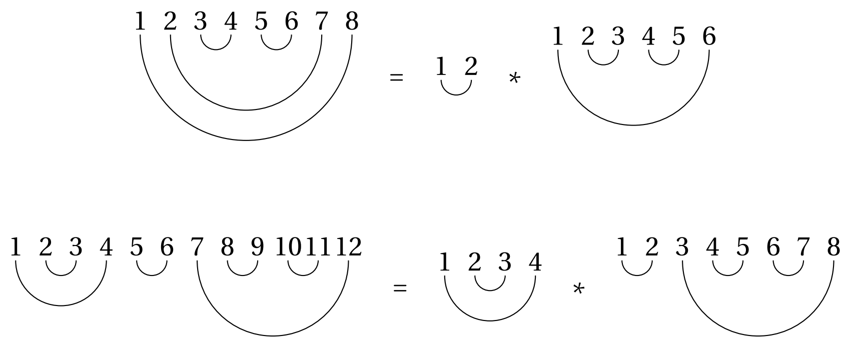
For U_2 we get an analogous trivialisation. Altogether, Y_{S_1} is a non-trivial fibre bundle with fibre a component of a smaller Springer fibre, again the one with cup diagram 

.

Definition 7.4. We define the *cup diagram decomposition* as follows:
If C is a cup diagram with all cups nested inside a single cup C', we say C is in nested position. If C is in nested position, then the cup diagram decomposition is $C = C' * D$ where D is C with C' removed and numbering adjusted.
If this is not the case, then the cup diagram decomposition is $C = D * D'$, where D consists of the cup B with $s(B) = 1$ and of all the cups nested inside B and D' consists of the remaining cups with the numbering adjusted.

For an extended cup diagram C let $|C|$ be twice the number of cups.

Example 7.5.



Definition 7.6. For n even, by Y_S^n we denote the irreducible component of the Springer fibre associated to a standard tableau S of type $(1,\ldots,n)$ and of shape $(\frac{n}{2},\frac{n}{2})$.

Note that we can associate an unique standard tableau S of type $(1,\ldots,n)$ and of shape $(\frac{n}{2},\frac{n}{2})$ to each cup diagram C consisting of black cups such that $eC(S) = C$ by writing the numbers at the right end of the cups in the bottom line and the other ones in the top line.

Lemma 7.7. *Let n be even and Y_S^n be a component of the Springer fibre such that $eC(S)$ is not in nested position and let $eC(S) = eC(R) * eC(T)$ its cup diagram decomposition. Let $r = |eC(R)|$ and $t = |eC(T)|$. Then Y_S^n is the total space of a trivial fibre bundle with base space Y_R^r and fibre Y_T^t.*

Proof. We have

$$Y_S^n = \{(F_1 \subset \cdots \subset F_r \subset F_{r+1} \subset \cdots \subset F_n) = F_\bullet | F_\bullet \text{ is } N\text{-invariant} \\ \text{and satisfy the conditions of } depG(S)\}$$

From the conditions of $depG(S)$ we know that $F_r = N^{-\frac{r}{2}}\{0\}$, thus it is a fixed subspace of $\mathbb{C}^n$. Now, we consider the map

$$\begin{array}{rcl} p: Y_S^n & \to & Y_R^r \\ (F_1 \subset \cdots \subset F_n) & \mapsto & \left(F_1 \subset \cdots \subset F_r = N^{-\frac{r}{2}}\{0\}\right). \end{array}$$

Analogous to Example 7.3 1) this is a trivial fibre bundle with fibre Y_T^t. □

Lemma 7.8. *Let n be even and Y_S^n be a component of the Springer fibre such that $eC(S)$ is in nested position and let $eC(S) = C * eC(T)$ its cup diagram decomposition. Let $t = |eC(T)|$. Then Y_S^n is the total space of a non-trivial fibre bundle with base space $\mathbb{CP}^1$ and fibre Y_T^t.*

Proof. Again, we consider the map

$$p: Y_S^n \to \mathbb{CP}^1$$
$$(F_1 \subset \cdots \subset F_n) \mapsto \left(F_1 \subset N^{-1}\{0\}\right).$$

As in Example 7.3 2) this is a non-trivial fibre bundle with fibre Y_T^t. □

Definition 7.9. We call an iterated fibre bundle of base type $(\underbrace{\mathbb{CP}^1, \ldots, \mathbb{CP}^1}_{l})$ an *l-iterated $\mathbb{CP}^1$-fibre bundle.*

Lemma 7.10. *Let $E = B \times F$ be a trivial fibre bundle and let B be an l_1-iterated $\mathbb{CP}^1$-fibre bundle and let F be an l_2-iterated $\mathbb{CP}^1$-fibre bundle. Then E is an (l_1+l_2)-iterated $\mathbb{CP}^1$-fibre bundle.*

Proof. Let $B = B_1, \ldots, B_{l_1}, B_{l_1+1} = pt$ be the total spaces of the iterated fibre bundles for B and let $F = F_1, \ldots, F_{l_2}, F_{l_2+1} = pt$ the ones for F. Then E is an iterated fibre bundle with total spaces $E, B_2 \times F, \ldots, B_{l_1} \times F, pt \times F = F, F_2, \ldots, F_{l_2}, pt$ and the associated maps. □

Proof of Theorem 7.2. First we assume $n = 2k$ and show the theorem for standard tableaux S of type $(1, \ldots, n)$, i.e. for irreducible components in the Springer fibre. We do induction on n:
If $k = 2$, then by Example 7.3 in both cases we get 2-iterated $\mathbb{CP}^1$-fibre bundles by including $\mathbb{CP}^1 \overset{\text{id}}{\to} \mathbb{CP}^1$ where the fibre is just one point.

Now, we consider Y_S with $|eC(S)| = 2k$. If $eC(S)$ is in nested position, then by Lemma 7.8 Y_S is the total space of a fibre bundle with base space $\mathbb{CP}^1$ and fibre Y_T with $|eC(T)| = 2k-2$, where $eC(T)$ arises from the cup diagram decomposition. By induction Y_T is a $k-1$-iterated $\mathbb{CP}^1$-fibre bundle, thus Y_S is a k-iterated $\mathbb{CP}^1$-fibre bundle.

In the other case, by Lemma 7.7 Y_S is the total space of a trivial fibre bundle with base space Y_R and fibre Y_T, where $|eC(R)|, |eC(T)| \leq 2k-2$ and $|eC(R)| + |eC(T)| = 2k$. By induction, Y_R and Y_T are $\frac{|eC(R)|}{2}$– iterated or $\frac{|eC(T)|}{2}$-iterated fibre bundles, respectively. Now the assertion follows from Lemma 7.10.

By the above for S a standard tableau of type $(1,\dots,n)$ and $n = 2k$ Y_S is an iterated fibre bundle, hence it is smooth as variety by [Har77, III.9,10]. By [Saf77, II.5] smooth varieties are normal. Since we are in characteristic zero, our morphism is separable. Thus, by a version of Zariski's main theorem [Pro07, 7.2] the bijective morphism of Proposition 5.30 is an isomorphism. Hence, $\mathcal{Y}_w$ for w a row strict tableau of type $(1,\dots,n)$ and shape $(n-k,k)$ with $n-k > k$ is also an iterated fibre bundle and smooth. As above, we conclude that the bijective morphism of Theorem 4.8 is an isomorphisms. Thus, $\widetilde{\mathcal{Y}}_w$ for w a row strict tableau of type $(i_1,\dots,i_m)$ is an iterated fibre bundle and smooth.

By Theorem 4.8 we can restrict ourselves to generalised irreducible components in Springer fibres, i.e. to those associated to cup diagrams without $\times\times$. By Proposition 5.30 it suffices to consider irreducible components associated to standard tableaux with equally long rows, i.e. those associated to cup diagrams with only black cups. Therefore, by the above discussion the assertion is shown for generalised irreducible components of Spaltenstein varieties. The assertion about the number of independents is transferred, since by Remark 5.22 the number of independents is given by the number of black cups and this stays the same in the reduction process.

Now we consider intersections of generalised irreducible components. Again, we only have to consider the ones with $CC(w,w')$ just consisting of black circles. This reduction is possible, since analogously as above with Theorem 4.8 we can restrict ourselves to intersections of generalised components in Springer fibres. Then we can delete the green circles by the analogon of Proposition 5.30. These analogons can be shown by computations similar to the ones above.

After that, we distinguish whether there is a circle in $CC(w,w')$ which contains all the others or not, and work with a decomposition of the circle diagrams analogously to the one of the cup diagrams above. In the case where there is a circle containing all the others, we get an analogon to Lemma 7.8 and in the other case one to Lemma 7.7. Then the assertion follows inductively as above. The assertion about the number of independents is also true, because by Remark 6.3 the number of independents is given by the number of black circles. □

Remark 7.11. As seen in the proof, the morphisms from Proposition 5.30 and Theorem 4.8 are isomorphisms of varieties.

8 Consequences and cohomology

In this section we use the result of the previous section to determine the dimension of irreducible components and to compute the cohomology of generalised irreducible components and their intersections. Then, we show that the cohomology of intersections can be computed by applying a functor from Part I to the associated circle diagram. We conclude this section by stating a conjecture which provides another connection to Part I.

Remark 8.1. For $F \to E \to B$ a fibre bundle with E, F, B manifolds, we get $\dim E = \dim F + \dim B$. In the proof of Theorem 7.2 we showed that the $\widetilde{\mathcal{Y}}_w$ are smooth as varieties, hence $(\widetilde{\mathcal{Y}}_w)^{an}$ or $(\widetilde{\mathcal{Y}}_w \cap \widetilde{\mathcal{Y}}_{w'})^{an}$ are complex manifolds.

Thus, by Theorem 7.2 we get that the dimension of $(\widetilde{\mathcal{Y}}_w)^{an}$ or $(\widetilde{\mathcal{Y}}_w \cap \widetilde{\mathcal{Y}}_{w'})^{an}$ is given by the number of independents in the associated dependence graph. Since by [Ser55] the dimension of X as variety coincides with the dimension of X^{an} as complex manifold, the same holds for $\widetilde{\mathcal{Y}}_w$ and $\widetilde{\mathcal{Y}}_w \cap \widetilde{\mathcal{Y}}_{w'}$. For $\widetilde{\mathcal{Y}}_w$ by Remark 5.22 the dimension coincides with the number of black cups in $eC(w)$.

In particular, this holds for $\widetilde{\mathcal{Y}}_S = \widetilde{Y}_S$. Thus, all the irreducible components of the Spaltenstein variety have the same dimension. This is true, because for a standard tableau S the number of black cups in $eC(S)$ is exactly the number of entries in $S_\vee \setminus S_\times$ which is always $k - \#S_\times$. For irreducible components of the Springer fibre this was already shown in [Spa76].

Alternatively, one can show the equality of the dimensions by using the statement for irreducible components of Springer fibres, Theorem 4.8 and [Hum75, Corollary4.3]. This is possible, since by the usual argumentation for standard tableaux one can show that the map p from Definition 4.7 sends standard tableaux to standard tableaux.

In the following, $H^*(X;\mathbb{C})$ denotes singular cohomology with coefficients in $\mathbb{C}$.

Definition 8.2. A space X is called of finite type, if $\dim_{\mathbb{C}} H^n(X;\mathbb{C})$ is finite for all n.

Lemma 8.3. *Let $(F \to E \to B)$ be a fibre bundle with F connected and of finite type, B of finite type, simply connected, paracompact and Hausdorff. Assume that $H^*(F;\mathbb{C})$ and $H^*(B;\mathbb{C})$ are concentrated in even degrees and $H^r(F;\mathbb{C}) = 0$ for $r \geq s$ and $H^r(B;\mathbb{C}) = 0$ for $r \geq t$.*
Then $H^*(E;\mathbb{C}) \cong H^*(B;\mathbb{C}) \otimes_{\mathbb{C}} H^*(F;\mathbb{C})$ as vector spaces. Furthermore, E is of finite type, $H^*(E;\mathbb{C})$ is concentrated in even degrees and $H^r(E;\mathbb{C}) = 0$ for $r \geq s + t$.

Proof. By [Spa66, §2.7] a fibre bundle with paracompact and Hausdorff base space is a fibration. The system of local coefficients is simple because $\pi_1(B) = 0$. Thus by [McC85, 5.5] we have $E_2^{p,q} \cong H^p(B;\mathbb{C}) \otimes_k H^q(F;\mathbb{C})$ in the Leray-Serre spectral sequence. Because of the concentration of the cohomologies in even degrees, the spectral sequence collapses at level 2 and we have $E_\infty^{p,q} \cong H^p(B;\mathbb{C}) \otimes_{\mathbb{C}} H^q(F;\mathbb{C})$. Since the spectral sequence converges to $H^*(E;\mathbb{C})$ and $\mathbb{C}$ is a field, we have

$$H^l(E;\mathbb{C}) \cong \bigoplus_{a+b=l} E_\infty^{a,b} \cong \bigoplus_{a+b=l} H^a(B;\mathbb{C}) \otimes_{\mathbb{C}} H^b(F;\mathbb{C}).$$

The rest follows from $H^*(E;\mathbb{C}) = \bigoplus_l H^l(E;\mathbb{C})$. □

In the following, we write $H^*(X)$ for $H^*(X^{an};\mathbb{C})$.

Corollary 8.4. *Let w, w' be row strict tableaux of type $(i_1, \dots, i_m)$ and assume $\widetilde{\mathcal{Y}}_w \cap \widetilde{\mathcal{Y}}_{w'} \neq \varnothing$. Then*

$$H^*(\widetilde{\mathcal{Y}}_w) \cong \big(\mathbb{C}[x]/(x^2)\big)^{\otimes u}$$
$$H^*(\widetilde{\mathcal{Y}}_w \cap \widetilde{\mathcal{Y}}_{w'}) \cong \big(\mathbb{C}[x]/(x^2)\big)^{\otimes v}$$

as vector spaces, where u and v are the number of independents.

Proof. By Theorem 7.2 we know that $(\widetilde{\mathcal{Y}}_w)^{an}$ and $(\widetilde{\mathcal{Y}}_w \cap \widetilde{\mathcal{Y}}_{w'})^{an}$ are iterated $\mathbb{CP}^1$-fibre bundles.

In each of the iterated fibre bundles the fibre is connected, since the fibres themselves are iterated fibre bundles, and if the fibre and the base space are connected, then the total space also is connected.

Furthermore, the base space is always the well-known space $\mathbb{CP}^1$. It is known that $\mathbb{CP}^1$ is simply connected and $H^*(\mathbb{CP}^1;\mathbb{C}) \cong \mathbb{C}[x]/(x^2)$, in particular $H^0(\mathbb{CP}^1;\mathbb{C}) = \mathbb{C} = H^2(\mathbb{CP}^1;\mathbb{C})$ and $H^l(\mathbb{CP}^1;\mathbb{C}) = 0$ for $l = 1$ or $l \geq 3$. Particularly, $H^*(\mathbb{CP}^1;\mathbb{C})$ is finitely generated and concentrated in even degrees. As a manifold $\mathbb{CP}^1$ is paracompact and Hausdorff.

Now, the claim inductively follows from Lemma 8.3 starting with $H^*(pt) = \mathbb{C}$, since the lemma also states that the conditions for the next step are fulfilled. Therefore, for the total space E of an l-iterated $\mathbb{CP}^1$-fibre bundle we have

$$H^*(E;\mathbb{C}) \cong \underbrace{\mathbb{C}[x]/(x^2) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{C}[x]/(x^2)}_{l} \otimes_{\mathbb{C}} \mathbb{C} \cong \big(\mathbb{C}[x]/(x^2)\big)^{\otimes l}.$$

□

Now we use the previous considerations to provide a connection to Part I by using the monoidal 2-functor F_{ColCob} to compute the cohomology.

Theorem 8.5 (Connection to F_{ColCob}).
The following diagram commutes:

$$\begin{tikzcd}
\textit{pairs of row strict tableaux} \arrow[r, "(w,w') \mapsto F_{ColCob}(CC(w,w'))"'] \arrow[d] & \textit{vector spaces} \arrow[d, "\mathrm{id}"] \\
\textit{pairs of } \widetilde{\mathcal{Y}}_w \arrow[r, "(\widetilde{\mathcal{Y}}_w, \widetilde{\mathcal{Y}}_{w'}) \mapsto H^*(\widetilde{\mathcal{Y}}_w \cap \widetilde{\mathcal{Y}}_{w'})"'] & \textit{vector spaces}
\end{tikzcd}$$

Proof. By Theorem 6.4 we have $H^*(\widetilde{\mathcal{Y}}_w \cap \widetilde{\mathcal{Y}}_{w'}) = 0$, if and only if $F_{ColCob}(CC(w, w')) = 0$. If $\widetilde{\mathcal{Y}}_w \cap \widetilde{\mathcal{Y}}_{w'} \neq \varnothing$, then the independents in $\widetilde{\mathcal{Y}}_w \cap \widetilde{\mathcal{Y}}_{w'}$ are exactly the leftmost points of the black connected components in $depG(w, w')$. By Remark 6.3 these are exactly the leftmost points in $CC(w, w')$, thus for each black circle we have one independent. Therefore, the theorem follows from Corollary 8.4. □

The above theorem gives a hint to the following conjecture:

Conjecture 8.6. *Mimicking the approach of [SW08, Theorem 35] there should be an associative convolution algebra structure on* $\bigoplus_{w,w'} H^*(\widetilde{\mathcal{Y}}_w \cap \widetilde{\mathcal{Y}}_{w'})$ *which can be described via the morphisms in Cob respectively ColCob.*

Appendix A

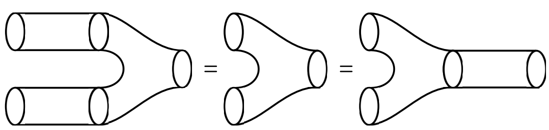
Relations

In this chapter we describe the relations of *Cob*, *ColCob* or *NesCob*. In addition to the pictured relations below we, furthermore, include all relations which arise by reflecting both sides of the displayed relations at the horizontal and/or the vertical axis. These should also belong to the relations of *Cob*, *ColCob* or *NesCob* respectively.

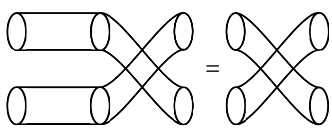
A.1 Relations for Cob

Identity relations:

(A.1.1)

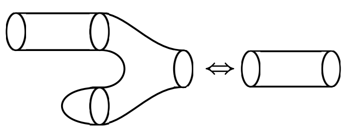


(A.1.2)



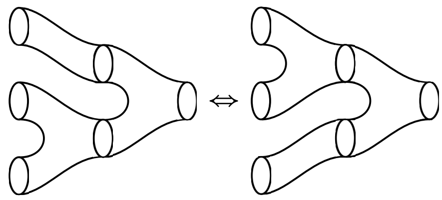
(A.1.3)

Unit:



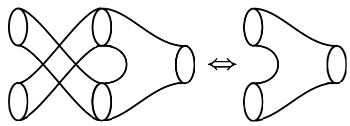
(A.1.4)

Associativity:



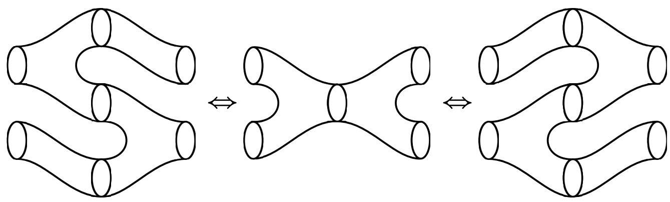
(A.1.5)

Commutativity:



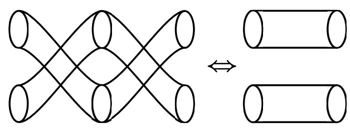
(A.1.6)

Frobenius relation:

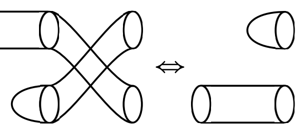


(A.1.7)

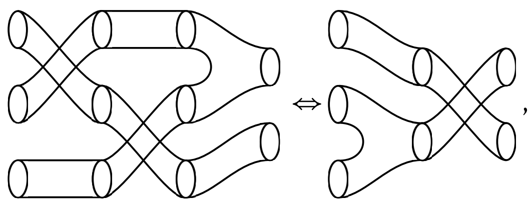
Twist relations:



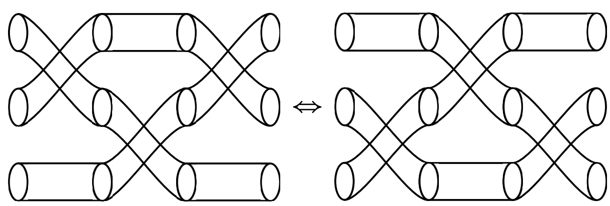
(A.1.8)



(A.1.9)



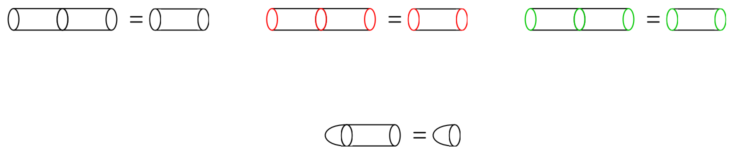
(A.1.10)

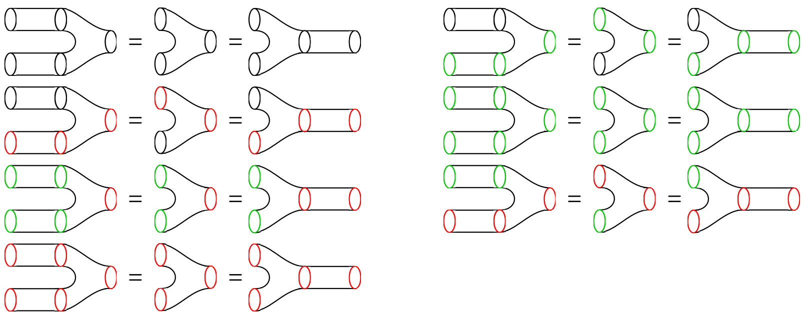


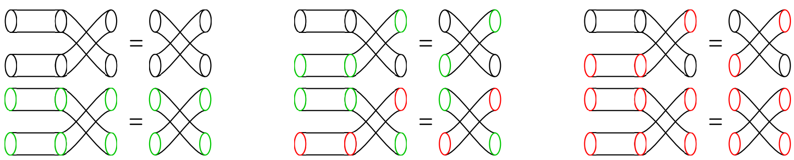
(A.1.11)

A.2 Relations for ColCob

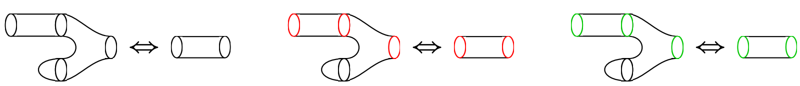
Coulered identities:



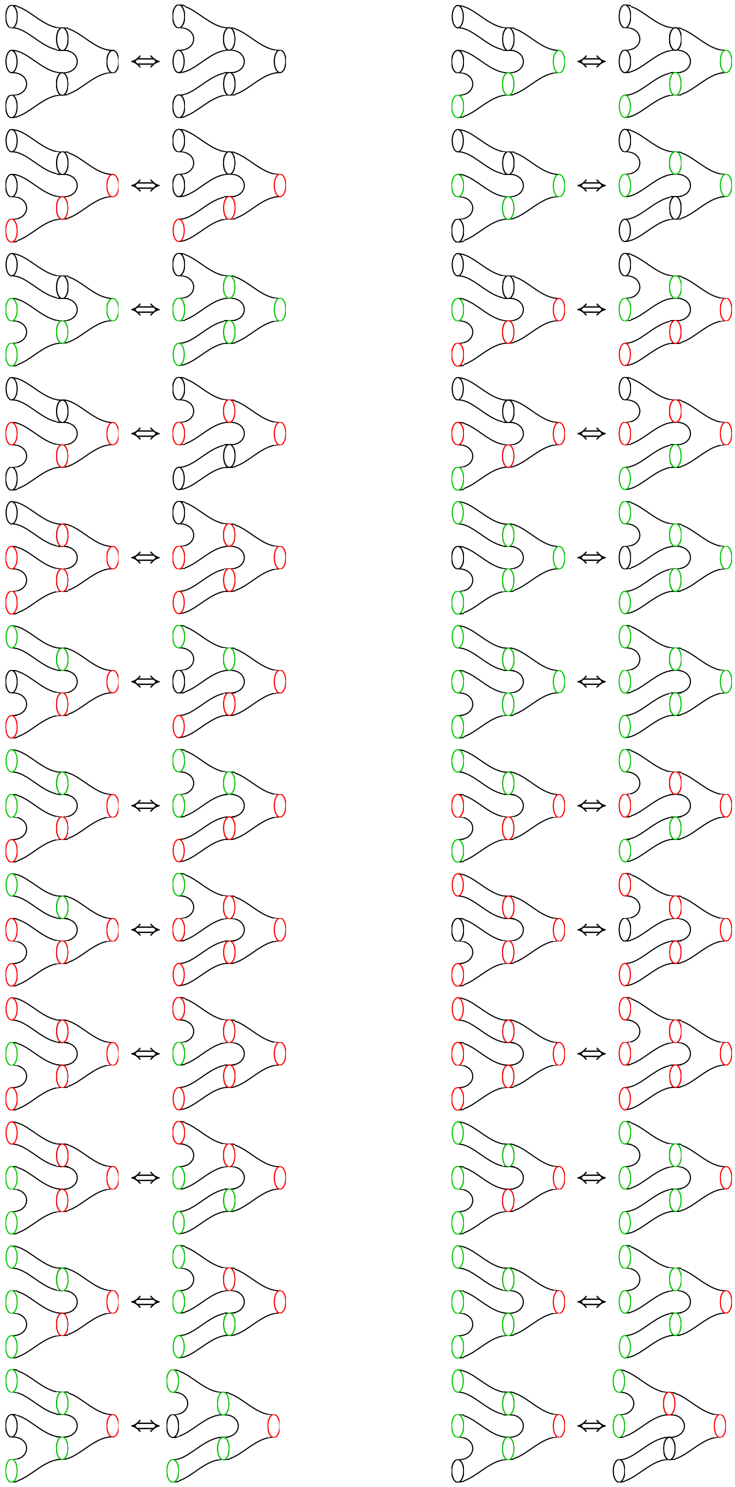




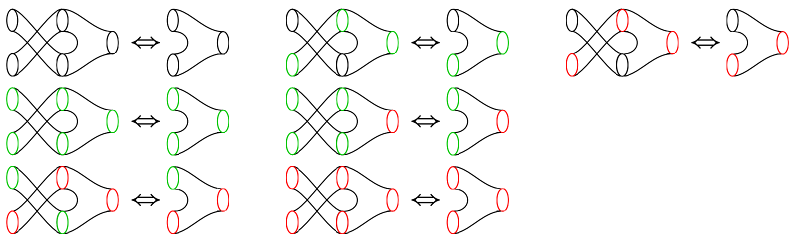
Unit:



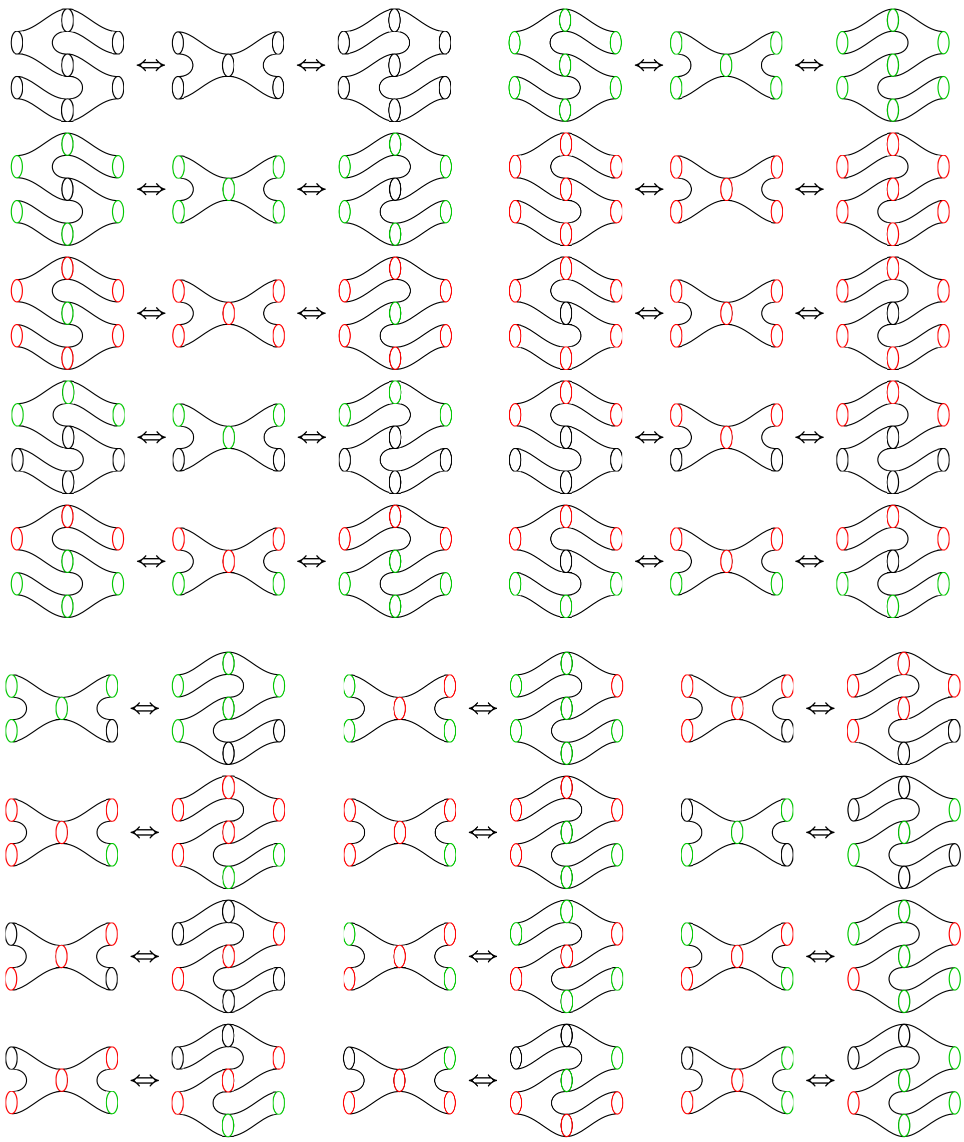
Associativity:



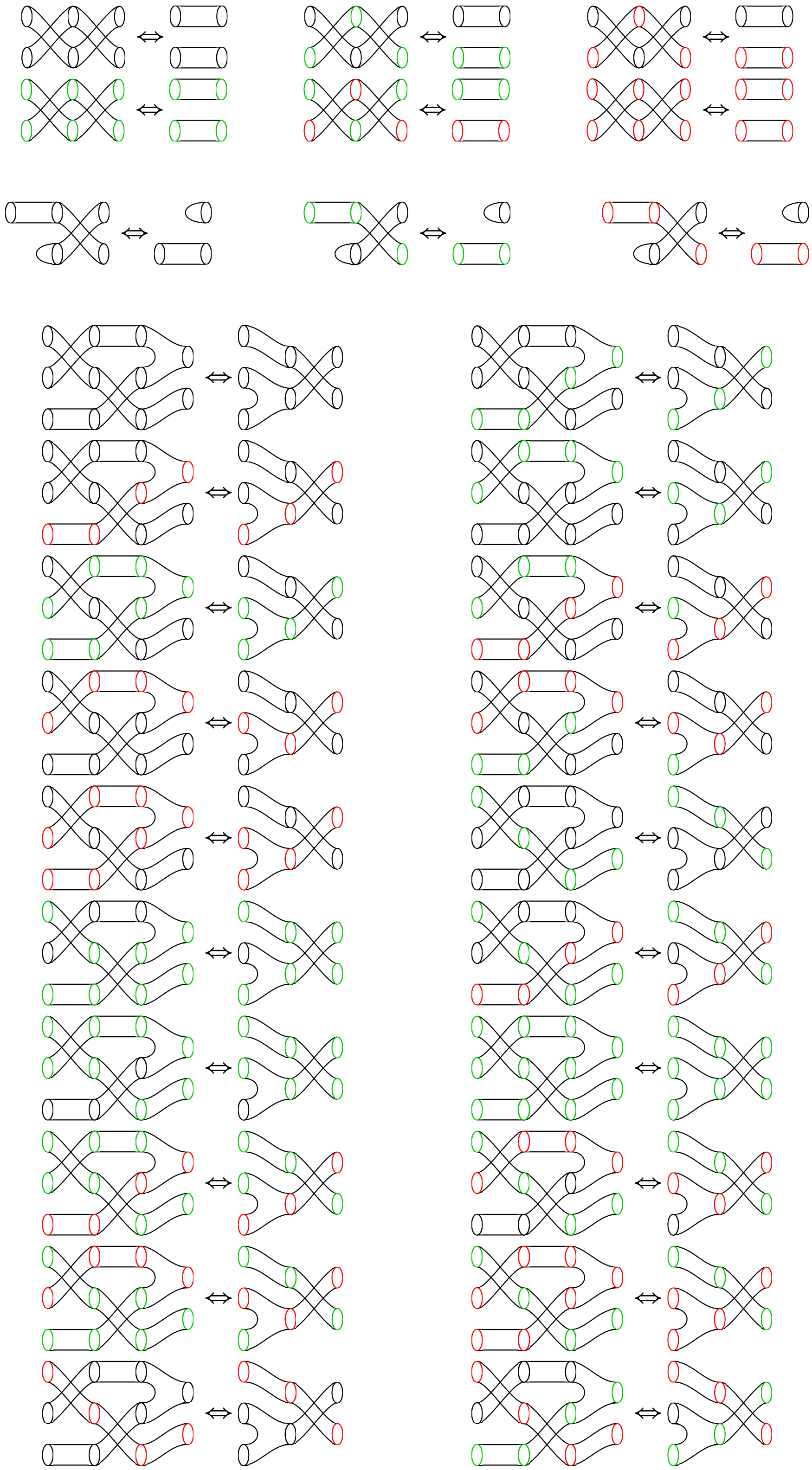
Commutativity:

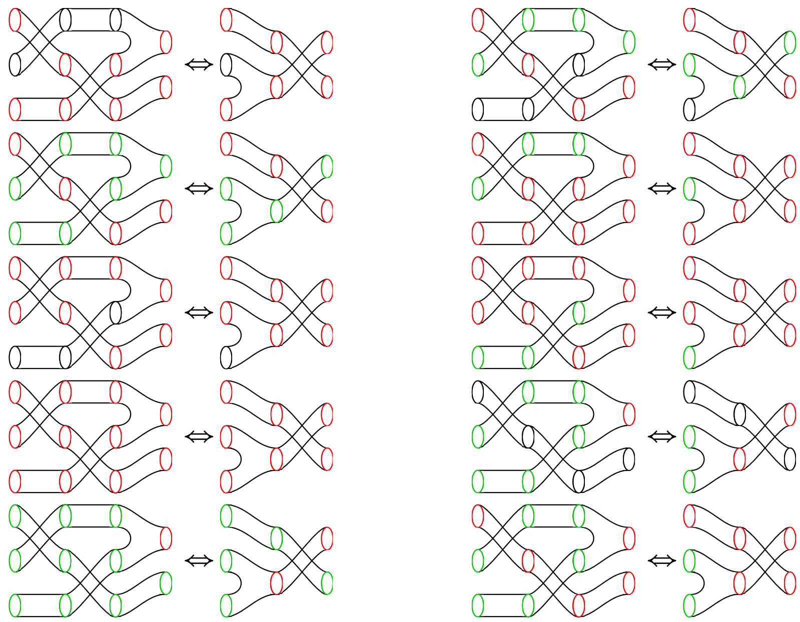


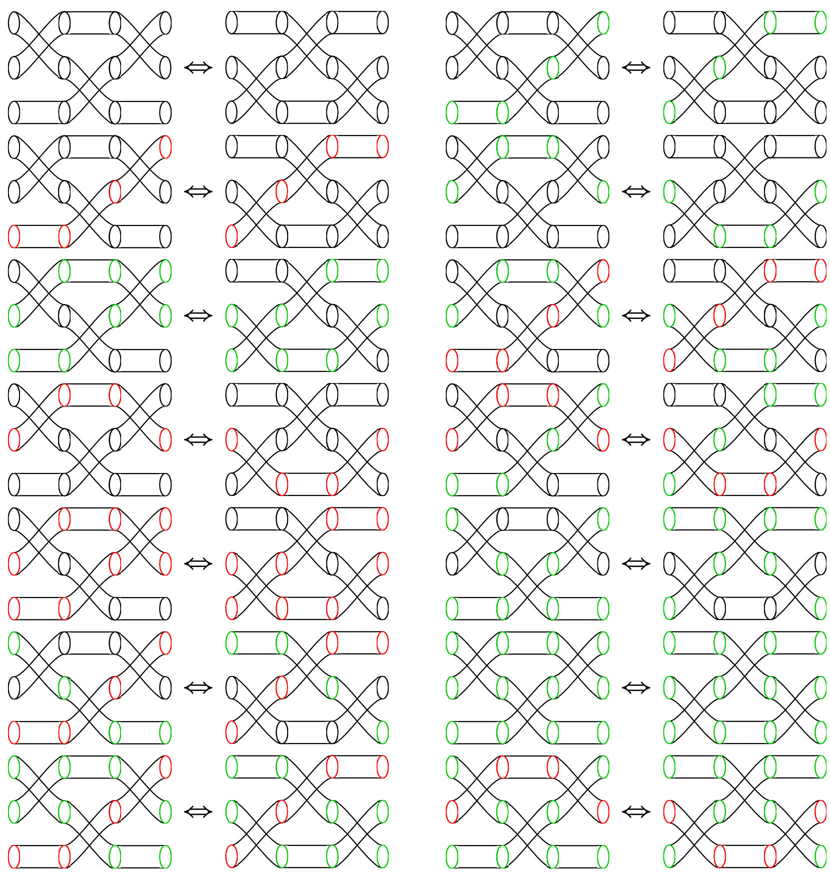
Generalised Frobenius relations:



Twist relations:







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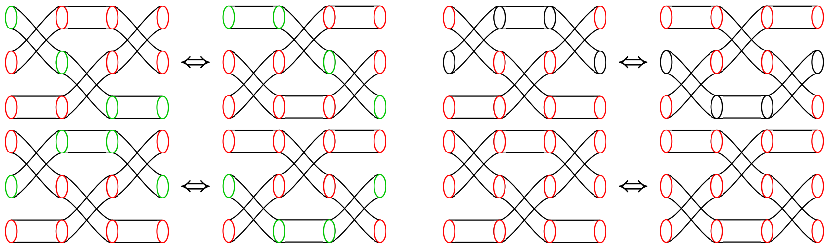
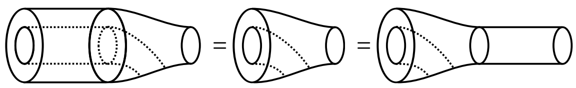


A.3 Relations for NesCob

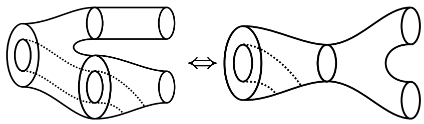
The relations for *NesCob* are those of *Cob* together with the following ones:

Identity relation:

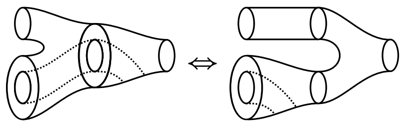


(A.3.1)

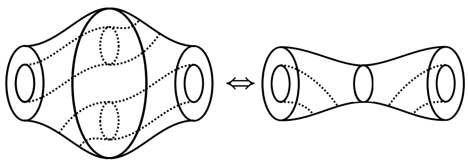
Further relations:



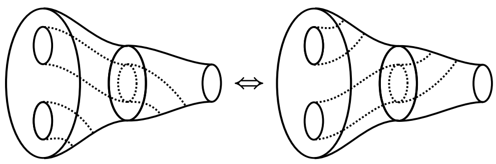
(A.3.2)



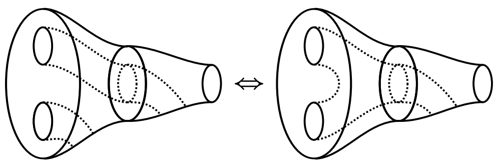
(A.3.3)



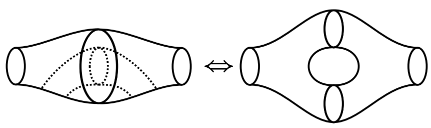
(A.3.4)



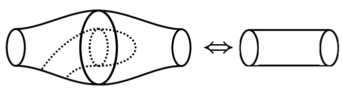
(A.3.5)



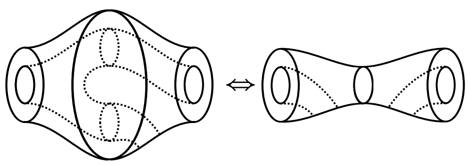
(A.3.6)



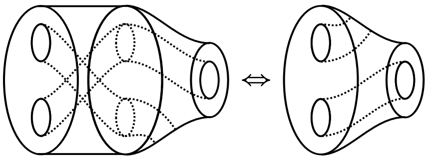
(A.3.7)



(A.3.8)

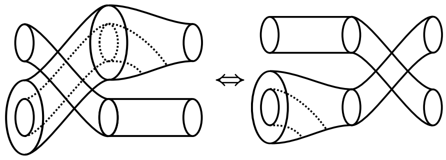


(A.3.9)



(A.3.10)

Twist relation:



(A.3.11)

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