On the Ext algebras of parabolic Verma modules and A_∞-structures

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Abstract

We study the Ext-algebra of the direct sum of all parabolic Verma modules in the principal block of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ for the hermitian symmetric pair $(\mathfrak{gl}_{n+m}, \mathfrak{gl}_n \oplus \mathfrak{gl}_m)$ and present the corresponding quiver with relations for the cases $n = 1, 2$. The Kazhdan-Lusztig combinatorics is used to deduce a general vanishing result for the higher multiplications in the A_∞-structure of a minimal model. An explicit calculations of the higher multiplications with non-vanishing m_3 is included.

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Introduction

In 1988 Shelton determined inductively the graded dimension of the spaces of extensions $\mathrm{Ext}^k(M(\lambda), M(\mu)) = \bigoplus_{k\geq 0} \mathrm{Ext}^k(M(\lambda), M(\mu))$ of parabolic Verma modules $M(\lambda)$ and $M(\mu)$ in the parabolic category $\mathcal{O}^{\mathfrak{p}}$ for the Hermitian symmetric cases [Sh]. More recently Biagioli reformulated the result combinatorially and obtained a closed dimension formula [Bi]. A nice feature is the fact that (parabolic) Verma modules form an exceptional sequence; i.e. they are labeled by a partially ordered set $(\Lambda, \leq)$ of highest weights such that for all $k \geq 0$ the following holds:

$$\mathrm{Hom}(M(\lambda), M(\lambda)) = \mathbb{C} \text{ and } \mathrm{Ext}^k(M(\lambda), M(\mu)) = 0 \text{ unless } \lambda \leq \mu.$$

A priori the set Λ is infinite, but the category $\mathcal{O}^{\mathfrak{p}}$ decomposes into indecomposable summands, so-called blocks, each containing only finitely many of the parabolic Verma modules. Taking M to be the direct sum of those which appear in the principal block yields a finite dimensional vector space $\mathrm{Ext}(M, M)$ which decomposes as the direct sum of $e_\mu \mathrm{Ext}(M, M) e_\lambda = \mathrm{Ext}(M(\mu), M(\lambda))$, where e_μ is the projection onto $M(\mu)$ along the sum of the other direct factors of M. It comes along with a natural algebra structure (the Yoneda product) which can be obtained by viewing $\mathrm{Ext}(M,M)$ as the homology of the algebra $\mathrm{Hom}(P_\bullet, P_\bullet)$ with $P_\bullet$ a projective resolution of M; the multiplication is given by the composition of maps between complexes. The construction of these projective resolutions and chain maps requires quite detailed knowledge of the projective modules and morphisms between them. Note that already the question about non-vanishing Hom-spaces between parabolic Verma modules is non-trivial (cf. [Bo] or [Hu, Theorem 9.10]). The aim of this paper is to explore this Ext-algebra in more detail for the Hermitian symmetric case of $(\mathfrak{gl}_{m+n}, \mathfrak{gl}_m \oplus \mathfrak{gl}_n)$. In [BS3] Brundan and the second author developed a combinatorial description of the category $\mathcal{O}^{\mathfrak{p}}$ for $\mathfrak{g} = \mathfrak{gl}_{m+n}$ and $\mathfrak{p}$ the parabolic subalgebra with Levi component $\mathfrak{gl}_m \oplus \mathfrak{gl}_n$ via a slight generalization of Khovanov's diagram algebra (cf. Theorem 3.1). Using these combinatorial techniques along with classical Lie theoretical results, provides enough tools to compute projective resolutions and their morphisms. As a crucial tool and byproduct we obtain a version of the Delorme-Schmid theorem (cf. [De], [Sc]) in our situation. The main results of the first part of the paper are Theorems 4.6 and 4.7, which give an explicit description of the Ext-algebra in terms of a path algebra of a quiver with relations for the cases $n = 1$ and $n = 2$, respectively. The first algebra also occurs in the context of knot Floer Homology, [KhSe], see also [AK]. For a connection to sutured Floer homology we refer to [GW].

In the context of Fukaya categories these algebras come along with a natural A_∞-algebra structure which encodes more information about the object. An A_∞-algebra, also known in topology as a strongly homotopic associative algebra, has higher multiplications satisfying so-called Stasheff relations (cf. [Ke]). As Keller for instance points out, working with minimal models provides the possibility to recover the algebra of complexes filtered by a family of modules $M(i)$ from some A_∞-structure on $\mathrm{Ext}(\bigoplus M(i), \bigoplus M(i))$. This A_∞-structure is constructed in the form of a minimal model, i.e. deduced from an algebra structure on $H^*(\mathrm{Hom}(\bigoplus P(i)_\bullet, \bigoplus P(i)_\bullet))$. In particular, there is a natural A_∞-structure on our space of extensions $\mathrm{Ext}(M,M)$. Since the projective objects are filtered by parabolic Verma modules and therefore parabolic Verma modules generate the bounded derived category $D^b(\mathcal{O}^{\mathfrak{p}})$ it is of interest to know more about these A_∞-structures. In the second part of the paper we construct an explicit minimal model for our Ext-algebra from above. The results from the first part, in particular the explicit construction of projective resolutions, allow us to analyse the higher multiplications. For the construction of the minimal models we mimic the approach of [LPWZ] and combine formulas obtained by Merkulov [Me] (for the case of superalgebras) and Kontsevich and Soibelman [KoSo] (for the $\mathbb{F}_2$-case). As for the Ext-algebra structure itself we keep track of all the signs (which sometimes leads to tedious computations). Using these techniques, we achieve the first vanishing theorem (Theorem 5.7) in case $n = 1$. In this theorem we get the formality of the Ext-algebra, i.e. we construct a minimal model with vanishing m_k for $k \geq 3$. For $n = 2$, in the second vanishing theorem (Theorem 5.9) we have an A_∞-structure with non-vanishing m_3 but vanishing m_k for $k \geq 4$. Thus, we obtain an example of an A_∞-algebra with non-trivial higher multiplications. The main result of the paper is presented in the general vanishing theorem (Theorem 5.6). It says that for arbitrary n we get a minimal model with vanishing m_k for $k \geq n^2 + 2$. A crucial ingredient in the proof is a detailed analysis of the Kazhdan-Lusztig polynomials forcing higher multiplications to vanish. This article is based on [Kl] and focuses on presenting the main results and techniques. Some of the (very) technical detailed calculations are therefore omitted, but can be found in [Kl].

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1. Preliminaries and Category $\mathcal{O}^{\mathfrak{p}}$

We first recall the definition of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$. For a more detailed treatment see [Hu], [MP].

Let $\mathfrak{g}$ be a finite dimensional reductive Lie algebra over $\mathbb{C}$ and $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ fixed Cartan and Borel subalgebras. Denote by $\Phi \subset \mathfrak{h}^*$ the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$ with the sets $\Delta \subset \Phi^+ \subset \Phi$ of simple and positive roots respectively. For $\alpha \in \Phi$ we have the root space $\mathfrak{g}_\alpha$ and the coroot $\alpha\check{} \in \mathfrak{h}$ normalized by $\alpha(\alpha\check{}) = 2$. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the triangular decomposition into negative roots spaces, Cartan subalgebra and positive root spaces. Denote $\Lambda^+ := \{\lambda \in \mathfrak{h}^* | \langle \lambda, \alpha\check{} \rangle \geq 0 \text{ for all } \alpha \in \Phi^+\}$, the set of dominant weights.

Denote by $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$ the half-sum of positive roots and by λ_0 the zero weight. Let W be the *Weyl group* with its usual *length function* $w \mapsto l(w)$ of taking the length of a reduced expression. We get a natural action of W on $\mathfrak{h}^*$ with fixed point zero. Shifting this fixed point to $-\rho$ defines the *dot-action* $w \cdot \lambda = w(\lambda + \rho) - \rho$. where $w \in W, \lambda \in \mathfrak{h}^*$.

For L any Lie algebra we denote by $U(L)$ the universal enveloping algebra. For $\lambda \in \mathfrak{h}^*$ and M an arbitrary $U(\mathfrak{g})$-module the *weight space* of weight λ relative to the action of the Cartan subalgebra $\mathfrak{h}$ is defined as

$$M_\lambda := \{v \in M \mid h \cdot v = \lambda(h)v, \ \forall \ h \ \in \ \mathfrak{h}\}.$$

We denote by $U(\mathfrak{g}) - \text{Mod}$ the category of left $U(\mathfrak{g})$-modules.

We fix now a standard parabolic subalgebra $\mathfrak{p}$ containing $\mathfrak{b}$. This corresponds to a choice of a subset $J \subset \Delta$ with associated root system $\Phi_J \subset \Phi$ such that $\mathfrak{p} = \mathfrak{l}_J \oplus \mathfrak{u}_J$ with nilradical $\mathfrak{u}_J$ and Levi subalgebra $\mathfrak{l}_J = \mathfrak{h} \oplus_{\alpha \in \Phi_J} \mathfrak{g}_\alpha$.

In particular, the choice $\mathfrak{p} = \mathfrak{b}$ corresponds to $J = \varnothing$ and $\mathfrak{l}_J = \mathfrak{h}$, whereas $\mathfrak{p} = \mathfrak{g}$ corresponds to $J = \Delta$ and $\mathfrak{l}_J = \mathfrak{g}$. Let $W_{\mathfrak{p}}$ be the Weyl group generated by all $\alpha \in J$. Denote by $W^{\mathfrak{p}}$ the set of minimal-length coset representatives for $W_{\mathfrak{p}} \backslash W$, that is

$$W^{\mathfrak{p}} = \{w \in W | \ \forall \ \alpha \in J : l(s_\alpha w) > l(w)\}.$$

Define the set of $\mathfrak{p}$-*dominant weights* as

$$\Lambda_J^+ := \{\lambda \in \mathfrak{h}^* | \langle \lambda, \alpha\check{} \rangle \in \mathbb{Z}^+ \text{ for all } \alpha \in J\}.$$

Denote by $E(\lambda)$ the finite dimensional $\mathfrak{l}_J$-module with highest weight $\lambda \in \Lambda_J^+$.

Definition 1.1. The category $\mathcal{O}^{\mathfrak{p}}$ is the full subcategory of $U(\mathfrak{g})-\mathrm{Mod}$ whose objects M satisfy the following conditions:

$\mathcal{O}1)$ M is a finitely generated $U(\mathfrak{g})$-module;

$\mathcal{O}2)$ M is $\mathfrak{h}$-semisimple, i.e., $M = \bigoplus_{\lambda\in\mathfrak{h}^*} M_\lambda$;

$\mathcal{O}3)$ M is locally $\mathfrak{p}$-finite, i.e. $\dim_{\mathbb{C}} U(\mathfrak{p})\cdot v < \infty$ for all $v \in M$.

We recall a few standard results on $\mathcal{O}^{\mathfrak{p}}$, see [Hu], [R-C] for details.

Definition 1.2. For $\lambda \in \Lambda_J^+$ we define the *parabolic Verma module*

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_J)} E(\lambda).$$

It has highest weight λ and is the largest quotient contained in $\mathcal{O}^{\mathfrak{p}}$ of the ordinary Verma module with highest weight λ. In particular, it has a unique simple quotient which is denoted by $L(\lambda)$. The $L(\lambda)$, for $\lambda \in \Lambda_J^+$ constitute a complete set of non-isomorphic simple objects in $\mathcal{O}^{\mathfrak{p}}$. The category $\mathcal{O}^{\mathfrak{p}}$ has enough projective objects; for $\lambda \in \Lambda_J^+$ let $P(\lambda)$ be the projective cover of $L(\lambda)$. The category $\mathcal{O}^{\mathfrak{p}}$ splits into direct summands (so-called 'blocks') $\mathcal{O}^{\mathfrak{p}}_\chi$,

$$\mathcal{O}^{\mathfrak{p}} = \bigoplus_{\chi} \mathcal{O}^{\mathfrak{p}}_\chi,$$

indexed by W-orbits under the dot-action. The summand $\mathcal{O}^{\mathfrak{p}}_\chi$ is the full subcategory of modules containing only composition factors of the form $L(\lambda)$ with $\lambda \in \chi \cap \Lambda_J^+$. In particular $M(\lambda)$ and $P(\lambda)$ are objects of $\mathcal{O}^{\mathfrak{p}}_\chi$ for $\lambda \in \chi$. Let $\mathcal{O}^{\mathfrak{p}}_0$ be the *principal block* of $\mathcal{O}^{\mathfrak{p}}$ corresponding to the orbit through zero which has precisely the $L(w\cdot\lambda_0)$ with $w \in W^{\mathfrak{p}}$ as simple objects. Since we work with left cosets, for better readability we write $P(x\cdot\lambda) =: P(\lambda.x)$; similarly for simple modules and parabolic Verma modules.

Remark 1.3. To combine later on Lie-theoretical results for $\mathcal{O}^{\mathfrak{p}}_0(\mathfrak{sl}_{m+n})$ with combinatorial results known for $\mathcal{O}^{\mathfrak{p}'}_0(\mathfrak{gl}_{m+n})$ we will tacitly use the standard equivalence of categories $\mathcal{O}^{\mathfrak{p}'}_0(\mathfrak{gl}_{m+n}) \cong \mathcal{O}^{\mathfrak{p}}_0(\mathfrak{sl}_{m+n})$ where $\mathfrak{p}'$ is the parabolic subalgebra with corresponding Levi component $\mathfrak{gl}_m \oplus \mathfrak{gl}_n$ and $\mathfrak{p} = \mathfrak{p}' \cap \mathfrak{sl}_{m+n}$.

2. The Ext algebra

We first introduce the homological and internal shift functors, $[i]$ and $\langle i\rangle$ for $i \in \mathbb{Z}$, on the category of complexes:

Convention 2.1. For a complex $C_\bullet = (C_\bullet, d_\bullet)$ define $C[i]_\bullet$ by $C[i]_j = C_{j-i}$ with differential $d[i]_j = (-1)^i d_j$. For M a graded A-module define the internal shift $M\langle i\rangle$ by $M\langle i\rangle_j = M_{j-i}$. We denote by $C_\bullet\langle i\rangle$ the (internally) shifted complex $C_\bullet$ obtained by just shifting each object; the differential maps stay homogeneous of degree zero.

Let $A, B \in Ob(\mathcal{A})$ be objects in an abelian category $\mathcal{A}$ and assume that A and B have finite projective dimension. Given projective resolutions $P_\bullet$ and $Q_\bullet$ of A and B, respectively, we define a differential graded structure on $\mathrm{Hom}(P_\bullet, Q_\bullet)$ with $\mathrm{Hom}(P_\bullet, Q_\bullet)^r = \prod_p \mathrm{Hom}(P_p, Q_{p+r})$ and differential $d_p(f) = d \circ f - (-1)^p f \circ d$ (c.f. [GM, III.6.13]). The space of extensions Ext can then be computed using the derived category,

$$\begin{aligned}
\mathrm{Ext}^k(A,B) &= \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(A[0], B[k]) && = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(P_\bullet[0], Q_\bullet[k]) \\
&= \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(P_\bullet[0], Q_\bullet[k]) && = \mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(P_\bullet, Q_\bullet)[k] \\
&= H^0(\mathrm{Hom}(P_\bullet, Q_\bullet)[k]) && = H^k(\mathrm{Hom}(P_\bullet, Q_\bullet)),
\end{aligned}$$

where the third equality holds because $P_\bullet$ is a bounded complex of projectives. In other words, $\mathrm{Ext}^k(A,B)$ can be determined by first computing the homomorphism spaces of the projective resolutions and afterwards taking its cohomology. Cycles in $\mathrm{Hom}(P_\bullet, Q_\bullet)$ are chain maps (according to the degree commuting or anticommuting) and boundaries are homotopies (up to sign). If considered as chain maps between translated complexes (i.e. in $\mathrm{Hom}_{\mathcal{D}^b(\mathcal{A})}(P_\bullet[0], Q_\bullet[k])$) with the sign convention 2.1, the cycles become commuting chain maps and boundaries stay usual homotopies.

We are now interested in the case $A = B$ and the algebra $\mathrm{Ext}^k(A,A) = H^k(\mathrm{Hom}(P_\bullet, P_\bullet))$. The multiplication in $\mathrm{Ext}(A,A)$ is induced from the multiplication in the algebra $\mathrm{Hom}(P_\bullet, P_\bullet)$, where it is given by composing of chain maps. Multiplication will be written from left to right, i.e. for α, $\beta \in \mathrm{Hom}(P_\bullet, P_\bullet)$ we have $(\alpha \cdot \beta)(x) = \beta(\alpha(x))$.

If $A = \bigoplus_{\alpha \in I} A_\alpha$ and $P_{\alpha\bullet}$ is a projective resolution of A_α with corresponding decomposition $P_\bullet = \bigoplus_{\alpha \in I} P_{\alpha\bullet}$ then $\mathrm{Id}_\alpha = [\mathrm{id}] \in \mathrm{Ext}^0(A_\alpha, A_\alpha)$. The elements Id_α form a system of mutual orthogonal idempotents, hence we can write

$$\mathrm{Ext}^k(A,A) = \bigoplus_{\alpha,\beta \in I} \mathrm{Id}_\alpha \mathrm{Ext}^k(A_\alpha, A_\beta) \mathrm{Id}_\beta \,.$$

It is then enough to determine $\mathrm{Ext}^k(A_\alpha, A_\beta)$ for any k, α, β and the products of elements $x \in \mathrm{Ext}^k(A_\alpha, A_\beta)$ and $y \in \mathrm{Ext}^l(A_\beta, A_\gamma)$, interpreting their product as

$$x \cdot y = \mathrm{Id}_\alpha \, x \, \mathrm{Id}_\beta \, \mathrm{Id}_\beta \, y \, \mathrm{Id}_\gamma \in \mathrm{Ext}^{k+l}(A,A).$$

3. $\mathcal{O}^{\mathfrak{p}}(\mathfrak{gl}_{m+n}(\mathbb{C}))$ via Khovanov's diagram algebra

We specialize now our setup to $\mathfrak{g} = \mathfrak{gl}_{m+n}(\mathbb{C})$ with the standard Borel subalgebra $\mathfrak{b}$ given by upper triangular matrices containing the Cartan $\mathfrak{h}$ of diagonal matrices. Let $\mathfrak{p}$ be the parabolic subalgebra associated to the Levi subalgebra $\mathfrak{l} = \mathfrak{gl}_m(\mathbb{C}) \oplus \mathfrak{gl}_n(\mathbb{C})$. Then our key tool is the following special case of the main theorem from [BS3], first observed in [St]:

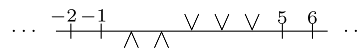


Figure 1: the zero weight for $n = 2$ and $m = 3$

Theorem 3.1. *There is an equivalence of categories from the principal block of $\mathcal{O}^{\mathfrak{p}}$ to the category of finite dimensional left modules over the Khovanov diagram algebra, $K_m^n - \mathrm{mod}$, sending the simple module $L(\lambda) \in \mathcal{O}^{\mathfrak{p}}$ to the simple module $L(\lambda) \in K_m^n - \mathrm{mod}$, the parabolic Verma module $M(\lambda)$ to the cell module $M(\lambda)$ and the indecomposable projectives to the corresponding indecomposable projectives.*

Here K_m^n is the algebra defined diagrammatically in [BS3] with an explicit distinguished basis given by certain diagrams (see below) and a multiplication defined by an explicit "surgery" construction which can be expressed in terms of an extended 2-dimensional TQFT construction, [St], generalizing a construction of Khovanov [Kh]. The distinguished basis is in fact a (graded) cellular basis in the sense of Graham and Lehrer [GL] in the graded version of Hu and Mathas [HM]. The algebra is shown to be quasi-hereditary in [BS1]. Hence we have cell or standard modules $M(\lambda)$, their projective covers $P(\lambda)$ and irreducible quotients $L(\lambda)$. This is meant by the notation used in the theorem.

3.1. The algebra K_m^n and its basic properties

For the construction of K_m^n, we recall from [BS1] the notions of weights, cup/cap/circle diagrams adapted to our situation. Let $\lambda \in \Lambda_J^+$ be the highest weight of a simple module in $\mathcal{O}_0^{\mathfrak{p}} = \mathcal{O}^{\mathfrak{p}}(\mathfrak{gl}_{m+n}(\mathbb{C})_0$ and let

$$\rho = \varepsilon_{m+n-1} + 2\varepsilon_{m+n-2} + \cdots + (m+n-1)\varepsilon_1 \in \mathfrak{h}^*.$$

The *(diagrammatical) weight* associated to λ is obtained by labeling the number i on the real line by $\vee$ if i belongs to $I_\vee(\lambda)$ and by $\wedge$ if i belongs to $I_\wedge(\lambda)$ respectively, where

$$\begin{aligned} I_\vee(\lambda) &:= \{(\lambda+\rho, \varepsilon_1), \ldots, (\lambda+\rho, \varepsilon_m)\} \\ I_\wedge(\lambda) &:= \{(\lambda+\rho, \varepsilon_{m+1}), \ldots, (\lambda+\rho, \varepsilon_{m+n})\}. \end{aligned}$$

Let Λ_m^n be the set of diagrammatical weights obtained in this way. Note that the labels are always on the $(m+n)$ places $i \in I = \{0, \ldots, m+n-1\}$ which we call *vertices*. The diagrammatical weight associated to λ_0 is given by putting all $\wedge$'s to the left and all $\vee$'s to the right, see Figure 1. In fact, Λ_m^n consists precisely of the diagrams obtained by permuting the n $\wedge$'s and m $\vee$'s establishing a bijection between the highest weights of parabolic Verma modules in $\mathcal{O}_0^{\mathfrak{p}}$ and elements in Λ_m^n. The dot-action corresponds then to permuting the labels; swapping $\vee$'s to the right means getting bigger in the Bruhat order, see [BS3, Section 1].

Figure 2: An oriented cup diagram and an oriented circle diagram.

We fix the above bijection and do not distinguish in notation between weights and diagrammatical weights. For $\lambda = \lambda_0.x$ with $x \in W^{\mathfrak{p}}$ we write $l(\lambda)$ for $l(x)$. For each $i \in I$ define *the relative length*

$$\begin{aligned} l_i(\lambda,\mu) := \quad & \#\{j \in I \mid j \leq i \text{ and vertex } j \text{ of } \lambda \text{ is labeled } \vee\} \\ & -\#\{j \in I \mid j \leq i \text{ and vertex } j \text{ of } \mu \text{ is labeled } \vee\} \end{aligned} \tag{3.1}$$

and note that $l(\lambda) - l(\mu) = \sum_{i \in I} \ell_i(\lambda,\mu)$ by [BS1, Section 5].

A *cup diagram* is a diagram obtained by attaching rays and finitely many cups (lower semicircles) to the vertices I, so that cups join two vertices $i \in I$, rays join vertices $i \in I$ down to infinity, and rays or cups do not intersect other rays or cups. A *cap diagram* is the horizontal mirror image of a cup diagram, so caps (i.e. upper semicircles) instead of cups are used. The mirror image of a cup (resp. cap) diagram c is denoted by c^*.

If c is a cup diagram and λ a weight in Λ_m^n, we can glue c and λ and obtain a new diagram denoted $c\lambda$. It is called an *oriented cup diagram* if

- each cup is oriented, i.e. one of its vertices is labeled $\vee$, and one $\wedge$;
- there are not two rays in c labeled $\vee\wedge$ in this order from left to right.

An example is given in Figure 2.

Similarly we can glue λ to a cap diagram c. The result λc is called *oriented cap diagram* if $c^*\lambda$ is an oriented cup diagram. A *circle diagram* is obtained by gluing a cup and a cap diagram at the vertices I. It consists of circles and lines. Gluing an oriented cap diagram with an oriented cup diagram along the same weight gives an *oriented circle diagram*. For an example, see Figure 2.

The *degree* of an oriented cup/cap diagram $a\lambda$ (or λb) means the total number of oriented cups (caps) that it contains. So in K_m^n one has $deg(a\lambda) \leq n$, since there are at most n cups. The *degree* of an oriented circle diagram $a\lambda b$ is defined as the sum of the degree of $a\lambda$ and the degree of λb. The *cup diagram associated to a weight* λ is the unique cup diagram $\underline{\lambda}$ such that $\underline{\lambda}\lambda$ is an oriented cup diagram of degree 0. (For an explicit construction: Take any two neighboring vertices labeled by $\vee\wedge$ and connect them by a cup. Repeat this procedure as long as possible, ignoring vertices which are already joined to others. Finally draw rays to all vertices which are left.) The *cap diagram associated to a weight* λ is defined as $\overline{\lambda} := (\underline{\lambda})^*$. The vector space underlying K_m^n has a basis

$$\{(a\lambda b) \mid \text{for all oriented circle diagrams with } \lambda \in \Lambda_m^n\}\ .$$

Each basis vector has a well-defined degree, turning the vector space into a graded vector space equipped with a distinguished homogeneous basis. The element e_λ is defined to be the diagram $\underline{\lambda}\lambda\overline{\lambda}$. The product of two circle diagrams $a\lambda b$ and $c\mu d$ is zero except for $b = c^*$. The multiplication of $a\lambda b$ and $b^*\mu d$ works by the rules of the generalized surgery procedure defined in [BS1, Section 3 and Theorem 6.1.]. The vectors $\{e_\alpha | \alpha \in \Lambda_m^n\}$ form a complete set of mutually orthogonal idempotents in K_m^n. We get

$$K_m^n = \bigoplus_{\alpha,\beta\in\Lambda_m^n} e_\alpha K_m^n e_\beta$$

where $e_\alpha K_m^n e_\beta$ has basis $\left\{(\underline{\alpha}\lambda\overline{\beta})\,|\lambda \in \Lambda_m^n \text{ such that the diagram is oriented}\right\}$.

3.2. Modules

Theorem 3.1 establishes an equivalence of categories between $\mathcal{O}_0^\mathfrak{p}$ and the category of finite dimensional K_m^n-modules. Following [BS1], we consider the category $K_m^n - \mathrm{gmod}$ of finite dimensional *graded* left K_m^n-modules which can be seen as a graded version of $\mathcal{O}_0^\mathfrak{p}$ with the following important objects:

- The *simple modules* $L(\lambda)$ with $\lambda \in \Lambda_m^n$.
  These are 1-dimensional modules concentrated in degree zero. The idempotent $e_\lambda \in K_m^n$ acts by the identity, all other e_μ by zero. Shifting the internal degree gives all simple objects, $L(\lambda)\langle i\rangle$, $i \in \mathbb{Z}$.

- The *projective cover* $P(\lambda) = K_m^n e_\lambda$ of the simple module $L(\lambda)$ has homogeneous basis

  $$\left\{(\underline{\alpha}\mu\overline{\lambda})\mid \text{for all } \alpha,\mu \in \Lambda_m^n \text{ such that the diagram is oriented}\right\};$$

  with the action induced from the diagrammatical multiplication in the algebra. By shifting the internal degree one obtains a full set of indecomposable graded projective modules.

- The *cell or standard modules* $M(\mu)$ with homogeneous basis

  $$\left\{(c\mu| \;\middle|\; \text{for all oriented cup diagrams } c\mu\right\}$$

  such that $(a\lambda b)(c\mu|) = (a\mu|)$ or 0 depending on the elements.

After forgetting the grading, these modules correspond via Corollary 3.1 to simple modules, projectives and Verma modules in the principal block of $O^\mathfrak{p}$.

3.3. q-decomposition numbers

We have the following theorems about cell module filtrations of projectives and Jordan-Hölder filtrations of cell modules, which say that K_m^n is quasi-hereditary in the sense of Cline, Parshall and Scott [CPS].

Lemma 3.2 ([BS1, Theorem 5.1]). *For $\lambda \in \Lambda_m^n$, enumerate the elements of the set $\{\mu \in \Lambda_m^n \mid \underline{\lambda}\mu$ is oriented$\}$ as $\mu_1, \mu_2, \ldots, \mu_n = \lambda$ so that $\mu_i > \mu_j$ implies $i < j$. Let $M(0) := \{0\}$ and for $i = 1, \ldots, n$ define $M(i)$ to be the subspace of $P(\lambda)$ generated by $M(i-1)$ and the vectors*

$$\left\{(c\mu_i\overline{\lambda}) \,\middle|\, \text{for all oriented cup diagrams } c\mu_i\right\}.$$

Then $M(0) \subset M(1) \subset \cdots \subset M(n) = P(\lambda)$ is a filtration of $P(\lambda)$ as graded K_m^n-module such that $M(i)/M(i-1) \cong M(\mu_i)\langle \deg(\mu_i\overline{\lambda})\rangle$ for $1 \le i \le n$.

Lemma 3.3 ([BS1, Theorem 5.2]). *For $\mu \in \Lambda_m^n$, let $N(j)$ be the submodule of $M(\mu)$ spanned by all graded pieces of degree $\ge j$. This defines a finite filtration of the graded K_m^n-module $M(\mu)$ with simple subquotients*

$$N(j)/N(j+1) \cong \bigoplus_{\substack{\lambda \subset \mu \text{ with} \\ \deg(\underline{\lambda}\mu) = j}} L(\lambda)\langle j\rangle.$$

By the BGG-reciprocity [Hu, Theorem 9.8(f)] the two multiplicities $d_{\lambda,\mu}^i := [M(\mu) : L(\lambda)\langle i\rangle]$ and $[P(\lambda) : M(\mu)\langle i\rangle]$ are equal and we get the symmetric *q-Cartan matrix*

$$C_{\Lambda_m^n}(q) = (c_{\lambda,\mu}(q))_{\lambda,\mu \in \Lambda_m^n},$$

where

$$c_{\lambda,\mu}(q) := \sum_{j \in \mathbb{Z}} \dim \operatorname{Hom}_{K_m^n}(P(\lambda), P(\mu))_j \, q^j \in \mathbb{Z}[q].$$

Set $d_{\lambda,\mu}(q) = \sum_i d_{\lambda,\mu}^i q^i$. Note that this sum in fact contains at most one non-trivial summand, since $d_{\lambda,\mu} \neq 0$ implies $\underline{\lambda}\mu$ is oriented and $\lambda \le \mu$ in the Bruhat ordering, in which case $d_{\lambda,\mu} = q^{\deg(\underline{\lambda}\mu)}$ holds (cf. [BS1, 5.12]).

In a cup (cap) diagram we number the cups (caps) $1, 2, \ldots$ according to their right vertex from left two right. For a cup (cap) diagram a we denote by $\operatorname{nes}_a(i)$ for $1 \le i \le \#\{\text{cups}\}$ the number of cups nested in the ith cup.

The following provides then explicit lower and upper bounds for the decomposition numbers and the entries of the q-Cartan matrix:

Proposition 3.4. *In $K_m^n - \mathrm{gmod}$ we have $d_{\lambda,\mu} = 0$ unless*

$$0 \le l(\lambda) - l(\mu) \le n + 2\sum_i \operatorname{nes}_{\underline{\lambda}}(i) \le n^2. \tag{3.2}$$

In particular, $c_{\lambda,\mu} = 0$ unless $l(\lambda) - l(\mu) \le n + 2\sum_i \operatorname{nes}_{\underline{\lambda}}(i) \le n^2$.

Proof. Assume $d_{\lambda,\mu}(q) \neq 0$. This means that $\underline{\lambda}\mu$ is oriented. By [BS1, Lemma 2.3] it follows that $\lambda \le \mu$ in the Bruhat ordering, which leads to $l(\lambda) \ge l(\mu)$. Now we find λ and μ such that $l(\lambda) - l(\mu)$ is maximal and $\underline{\lambda}\mu$ is oriented. Fix such λ and consider weights μ of smallest possible length such that $\underline{\lambda}\mu$ is still oriented. This is obtained if all $\wedge$'s and $\vee$'s on the end of a cup in λ are interchanged. Since a $\wedge$ on the ith cup has been moved $1 + 2\text{nes}_{\underline{\lambda}}(i)$ positions to the right, the length is changed by $\sum_i (2\text{nes}_{\underline{\lambda}}(i) + 1)$. Therefore, we obtain

$$0 \leq l(\lambda) - l(\mu) \leq n + 2 \sum_i \text{nes}_{\underline{\lambda}}(i).$$

Since $\sum_i \text{nes}_a(i)$ is maximal if all cups are nested (i.e if the jth cup contains precisely $j-1$ cups). In that case we obtain

$$2 \sum_i \text{nes}_a(i) = 2 \sum_{i=1}^{n} (i-1) = (n-1)n$$

and therefore (3.2) holds. For $c_{\lambda,\mu} \neq 0$ a simple $L(\lambda)$ must occur in $P(\mu)$, especially it must occur in some $M(\nu)$, i.e. $d_{\lambda,\nu} \neq 0$ and $d_{\mu,\nu} \neq 0$. Therefore,

$$l(\lambda) - l(\nu) \leq n + 2 \sum_i \text{nes}_{\underline{\lambda}}(i)$$

and $0 \leq l(\mu) - l(\nu)$ which implies

$$l(\lambda) - l(\mu) \leq l(\lambda) - l(\nu) \leq n + 2 \sum_i \text{nes}_{\underline{\lambda}}(i),$$

which proves the second inequality. □

3.4. Linear projective resolutions of cell modules

To compute the Ext-algebras of Verma modules it will be useful to construct explicitly linear projective resolutions of the cell modules $M(\lambda) \in K_m^n - \text{gmod}$. Recall that a projective resolution $P_\bullet$ is *linear* if P_i is generated by its homogeneous component in degree i. From the description of projective modules it is clear that $\bigoplus_{\lambda \in \Lambda_m^n} P(\lambda) \cong K_m^n$ is a minimal projective generator of $K_m^n - \text{mod}$. Any endomorphism is given by right multiplication with an element of the algebra, and $\text{Hom}_{K_m^n}(P(\lambda), P(\mu)) = \text{Hom}_{K_m^n}(K_m^n e_\lambda, K_m^n e_\mu) = e_\lambda K_m^n e_\mu$ as vector spaces, [BS1, (5.9)].

To construct the differentials in linear projective resolutions, we study first the degree 1 component of $\text{Hom}_{K_m^n}(P(\lambda), P(\mu))$, i.e. we search for elements ν s.t. $\deg(\underline{\lambda}\nu\overline{\mu}) = 1$. Since $1 = \deg(\underline{\lambda}\nu\overline{\mu}) = \deg(\underline{\lambda}\nu) + \deg(\nu\overline{\mu})$, one summand has to be 0 and the other 1.

1. $\deg(\underline{\lambda}\nu) = 0$, i.e. $\lambda = \nu$, so we look for an oriented cap diagram $\lambda\overline{\mu}$ of degree 1. It exists iff $\lambda > \mu$ and $\mu = \lambda.w$ with w changing the $\wedge$ and $\vee$ (in this ordering) at the end of a cup into a $\vee$ and $\wedge$.
2. $\deg(\nu\overline{\mu}) = 0$, i.e. $\mu = \nu$, so we look for an oriented cup diagram $\underline{\lambda}\mu$ of degree 1. It exists iff $\mu > \lambda$ and $\lambda = \mu.w$ with w changing the $\vee$ and $\wedge$ at the end of a cap.

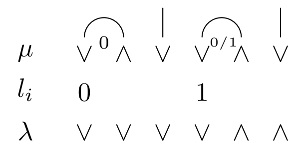


Figure 3: the labeled cap diagram

Altogether we get dim $\mathrm{Hom}_{K_m^n}(P(\lambda), P(\mu))_1 \leq 1$ and the diagram calculus defines a distinguished morphism $f_{\lambda,\mu}$ in case this dimension equals 1.

On the other hand, the modules occurring in a linear projective resolution of cell modules are determined by polynomials $p_{\lambda,\mu}$ defined diagrammatically and recursively in [BS2, Lemma 5.2.], namely certain Kazhdan-Lusztig polynomials going back to work of Lascoux and Schützenberger [LS].

We recall the construction of these polynomials. Set $p_{\lambda,\mu} = 0$ if $\lambda \not\leq \mu$. A *labeled cap diagram* C is a cap diagram whose unbounded chambers are labeled by zero and given two chambers separated by a cap, the label in the inside chamber is greater than or equal to the label in the outside chamber.

Definition 3.5. Denote by $D(\lambda, \mu)$ the set of all labeled cap diagrams obtained by labeling the chambers of $\overline{\mu}$ in such a way that for every inner cap c (a cap containing no smaller one), the label l inside c satisfies $l \leq l_i(\lambda, \mu)$, where i denotes the vertex of c labeled by $\vee$. The polynomials are given by

$$p_{\lambda,\mu}(q) := \sum_i p_{\lambda,\mu}^{(i)} q^i := q^{l(\lambda)-l(\mu)} \sum_{C \in D(\lambda,\mu)} q^{-2|C|}. \tag{3.3}$$

where $|C|$ denotes the sum of all labels in C.

Example 3.6. Figure 3 presents the possible labeled cap diagrams from $D(\lambda, \mu)$ for the chosen λ and μ. Since $l(\lambda) - l(\mu) = 4$, we get $p_{\lambda,\mu}(q) = q^4 + q^2$.

Theorem 3.7 ([BS2, Theorem 5.3], [Kl, Theorem 3.20])**.** *For $\lambda \in \Lambda_m^n$ the cell module $M(\lambda)$ has a linear projective resolution $P_\bullet(\lambda)$ of the form*

$$\cdots \xrightarrow{d_1} P_1(\lambda) \xrightarrow{d_0} P_0(\lambda) \xrightarrow{\varepsilon} M(\lambda) \longrightarrow 0 \tag{3.4}$$

with $P_0(\lambda) = P(\lambda)$ and $P_i(\lambda) = \bigoplus_{\mu \in \Lambda_m^n} p_{\lambda,\mu}^{(i)} P(\mu)\langle i \rangle$ for $i \geq 0$.

Using the above observations and tools from the proof of [BS2, Theorem 5.3], [Kl, §3.3.3] gives an explicit method to construct projective resolutions of cell modules in K_m^n − gmod by an interesting simultaneous induction varying the underlying algebra and the highest weights. For K_m^0 and K_0^n we have, up to isomorphism, only one indecomposable module, which is projective, simple and cell module at once. This provides the starting point of the induction. In the following we will fix such a projective resolution $P_\bullet(\lambda)$ for each λ. Together with the inequalities obtained before, we can deduce:

Proposition 3.8. *If a projective module $P(\nu)$ occurs as a direct summand in $P_i(\lambda)$ with $P_\bullet(\lambda)$ being the projective resolution constructed above, one has*

$$l(\lambda) - i - \left(n^2 - n - 2\sum_i \mathrm{nes}_{\underline{\nu}}(i)\right) \leq l(\nu) \leq l(\lambda) - i.$$

Proof. Let C be a cap connected with the jth $\wedge$ occurring in $\overline{\nu}$ and let it be the k_jth cup in our numbering with starting point i. Recall from (3.1) that $l_i(\lambda,\nu) \leq \{k|\, k \leq i \text{ and vertex } k \text{ of } \nu \text{ is labeled } \wedge\}$, the latter counting the numbers of $\wedge$'s to the left of the cap. This equals $j - 1 - \mathrm{nes}_{\underline{\nu}}(k_j)$ counting to ones the left of the jth $\wedge$ without those lying inside the cap, and thus

$$0 \leq |C| \quad \leq \sum_{\substack{j\in\{1,\ldots n\} \\ \text{cap ending on } j\text{th } \wedge}} (j - 1 - \mathrm{nes}_{\underline{\nu}}(k_j)) \leq \frac{n(n-1)}{2} - \sum_i \mathrm{nes}_{\underline{\nu}}(i).$$

If a module $P(\nu)$ occurs in the resolution (say at homological degree i), one has $p^{(i)}_{\lambda,\nu} > 0$, i.e. there is a diagram C such that $i = l(\lambda) - l(\nu) - 2|C|$. Taking the upper and lower bound for C obtained before, one gets

$$l(\lambda) - i - (n^2 - n - 2\sum_i \mathrm{nes}_{\underline{\nu}}(i)) \leq l(\nu) \leq l(\lambda) - i$$

and the claim of the proposition follows. □

The following is a vanishing result for $\mathrm{Ext}^k(M(\lambda), M(\mu))$:

Lemma 3.9. *For λ, $\mu \in \Lambda^n_m$ we have*

$$\mathrm{Hom}^k(P_\bullet(\lambda), P_\bullet(\mu)) = 0 \qquad \textit{unless } l(\lambda) \leq l(\mu) + n^2 + k. \tag{3.5}$$

Proof. A map between $P_\bullet(\lambda)$ and $P_\bullet(\mu)[k]$ is in each component a morphism between graded projective modules. Including the shift we therefore have to consider morphisms between projectives $P(\nu)$ occurring in $P_i(\lambda)$ and projectives $P(\nu')$ in $P_{i-k}(\mu)$. By Proposition 3.8 we know

$$l(\lambda) - i - \left(n^2 - n - 2\sum_i \mathrm{nes}_{\underline{\nu}}(i)\right) \leq l(\nu) \quad \textit{and} \quad l(\nu') \leq l(\mu) - (i-k).$$

Therefore, we have

$$l(\lambda) - i - \left(n^2 - n - 2\sum_i \mathrm{nes}_{\underline{\nu}}(i)\right) - (l(\mu) - (i-k)) \leq l(\nu) - l(\nu'). \tag{3.6}$$

Since we have a morphism between these projectives we get from Lemma 3.4

$$l(\nu) - l(\nu') \leq n + 2\sum_i \mathrm{nes}_{\underline{\nu}}(i). \tag{3.7}$$

Combining the two inequalities (3.6) and (3.7), we obtain

$$l(\lambda) - i - \left(n^2 - n - 2\sum_i \operatorname{nes}_{\underline{\nu}}(i)\right) - (l(\mu) - (i-k)) \leq n + 2\sum_i \operatorname{nes}_{\underline{\nu}}(i), \quad (3.8)$$

which implies $l(\lambda) \leq l(\mu) + n^2 + k$. The claim follows. □

4. The Ext-algebra of $\bigoplus_{x \in W^{\mathfrak{p}}} M(\lambda_0 \cdot x)$

Assume we are in the setup of Section 3 and denote

$$E^n_m = \bigoplus_{x,y \in W^{\mathfrak{p}}} \operatorname{Ext}\left(M(x \cdot \lambda_0), M(y \cdot \lambda_0)\right) = \bigoplus_{\lambda,\mu \in \Lambda^n_m} \operatorname{Ext}_{K^n_m}\left(M(\lambda), M(\mu)\right).$$

A very useful tool for describing E^n_m are Shelton's recursive dimension formulas which he established in [Sh] more generally for all the hermitian symmetric cases. For an arbitrary parabolic subalgebra $\mathfrak{p}$, there is no explicit formula, not even a candidate.

Abbreviating $E^k(x,y) = \dim \operatorname{Ext}^k(M(\lambda_0.x), M(\lambda_0.y))$ for $x, y \in W^{\mathfrak{p}}$, [Sh, Theorem 1.3] can be formulated as follows:

Theorem 4.1 (Dimension of Ext-spaces)**.** *With $\mathfrak{g}$ and $\mathfrak{p}$ as above, let $x, y \in W^{\mathfrak{p}}$ and let s be a simple reflection with $x > xs$ and $xs \in W^{\mathfrak{p}}$. The dimensions $E^k(x,y)$ are then given by the following formulas:*

1. $E^k(x,y) = 0 \qquad \forall\ k$ *unless* $y \leq x$;
2. $E^k(x,x) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{otherwise.} \end{cases}$

For $y < x$ there are the following recursion formulas:

3. $E^k(x,y) = E^k(xs, ys) \qquad$ *if $y > ys$ and $ys \in W^{\mathfrak{p}}$;*
4. $E^k(x,y) = E^{k-1}(xs, y) \qquad$ *if $ys \notin W^{\mathfrak{p}}$;*
5. $E^k(x,y) = E^{k-1}(xs, y) + E^k(xs, y) \qquad$ *if $ys > y$ but $xs \not> ys$;*
6. $E^k(x,y) = E^{k-1}(xs, y) - E^{k+1}(xs, y) + E^k(xs, ys) \qquad$ *if $x > xs > ys > y$.*

To translate between our setup and Shelton's note that he denotes $N_y = M(\lambda_0.\omega_{\mathfrak{m}} y \omega_0)$ where ω_0 and $\omega_{\mathfrak{m}}$ are the longest elements in W and in $W_{\mathfrak{p}}$ respectively. Then it only remains to observe that for $y, x \in W$ we have $\omega_{\mathfrak{m}} y \omega_0 \in W^{\mathfrak{p}} \Leftrightarrow y \in W^{\mathfrak{p}}$ and $\omega_{\mathfrak{m}} y \omega_0 < \omega_{\mathfrak{m}} x \omega_0 \Leftrightarrow y > x$ in the Bruhat order.

Although the previous theorem determines all dimension of Ext-spaces, it is convenient to have explicit vanishing conditions. Therefore, we reprove the Delorme-Schmid Theorem (cf. [De], [Sc]) in our situation:

Lemma 4.2. *For $\lambda, \mu \in \Lambda_m^n$ we have*

$$\mathrm{Ext}^k(M(\lambda), M(\mu)) = 0 \qquad \forall\ k > l(\lambda) - l(\mu).$$

Proof. We claim that any chain map $f : P_\bullet(\lambda) \to P_\bullet(\mu)[k]$ with $k > l(\lambda) - l(\mu)$ is homotopic to zero. On the kth component f induces a map $f_k : P_k(\lambda) \to P_0(\mu) = P(\mu)$. For $P(\nu)$ occurring as a direct summand in $P_k(\lambda)$ we have $l(\nu) \le l(\lambda) - k < l(\lambda) - (l(\lambda) - l(\mu)) = l(\mu)$ by Lemma 3.8. By Proposition 3.4 $L(\nu)$ does not occur in $M(\mu)$ and so the composition $P(\nu) \to P(\mu) \to M(\mu)$ is zero. Let $P_\bullet^T(\lambda)$ be the truncated complex with $P_i^T(\lambda) = 0$ for $i < 0$ and $P_i^T(\lambda) = P_{i+k}(\lambda)$ if $i \ge 0$. This is a projective resolution of $\operatorname{im} d_k$, and $f_\bullet$ induces a morphism $\widetilde{f}_\bullet : P_\bullet^T(\lambda) \to P_\bullet(\mu)$ such that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \cdots & \longrightarrow & P_0^T(\lambda) & \longrightarrow & \operatorname{im} d_k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \widetilde{f}_0} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \cdots & \longrightarrow & P(\mu) & \longrightarrow & M(\mu) & \longrightarrow & 0
\end{array}$$

where $\widetilde{f}$ is a lift of the zero map. Since the zero map between the complexes is also a lift of the zero map and two lifts are equal up to homotopy ([GM, Theorem III.1.3]) the map $\widetilde{f}$ is nullhomotopic by a homotopy $H : P_\bullet^T(\lambda) \to P_\bullet(\mu)[-1]$. This extends to a homotopy $H : P_\bullet(\lambda) \to P_\bullet(\mu)[-1]$ by defining it to be zero on the other terms. The claim follows. □

Remark 4.3. The result of Lemma 4.2 could also be deduced from Shelton's formulas or from the explicit formulas [Bi, Theorem 3.4].

Now we want to describe the Ext-algebra in the cases $(m, n) = (1, N)$ and $(m, n) = (2, N-1)$. The first algebra is related to algebras appearing in (knot) Floer homology, see [KhSe], [GW], the second invokes our theory in a more substantial way and provides interesting A_∞-structures.

Using the above tools, one can construct explicit maps between the projective resolutions from Theorem 3.7 and determine their linear dependence up to null homotopies. In this way we will obtain certain non-trivial elements in Ext^i which, using Shelton's dimension formulas, can be shown form a basis. Finally we compute the multiplication rules. Especially in the case for $n = 2$ the computations are long and cumbersome and carried out in [Kl]. We present the arguments and details for the case $n = 1$ here, the results and main idea for $n = 2$, and refer to [Kl] for the details.

4.1. The case $n = 1$

The elements in $W^{\mathfrak{p}}$ are precisely $s_1 \cdots s_j$, $0 \le j \le N-1$ and we abbreviate $(j) = \lambda_0.s_1 s_2 \ldots s_j$. The filtrations in Theorems 3.2 and 3.3 combined determine the filtration of projective modules in terms of simple modules; the Kazhdan-Lusztig polynomials for $(s) = \mu \ge \lambda = (j)$ are $p_{\lambda,\mu} = q^{j-s}$, see Figure 4. By

$\lambda=(j)$	$P(j)$
$j\neq 0$ $j\neq N$	$L(j)$ $L(j+1)L(j-1)$ $L(j)$
$j=0$	$L(0)$ $L(1)$
$j=N$	$L(N)$ $L(N-1)$ $L(N)$

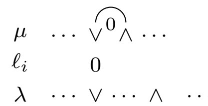


Figure 4: Jordan-Hölder series of $P(\lambda)$ (the colours indicate the Verma module)

Theorem 3.7 there is then a unique summand occurring in the ith position of the linear projective resolution of $M(\lambda)$, namely the projective module $P(j-i)$, and we have the distinguished morphism $f_k := f_{k,k+1}$, homogeneous of degree 1, from $P(k)$ to $P(k+1)$. Set $\mathrm{d}_{n-k}(n) = (-1)^{n+k+1} f_k$.

Lemma 4.4. *The chain complex*

$$0 \to P(0)\langle n\rangle \stackrel{d_0}{\to} P(1)\langle n-1\rangle \to \cdots \stackrel{d_{n-1}}{\to} P(n) \to 0$$

is a (linear) projective resolution of $M(n)$ in $K_N^1-\mathrm{gmod}$.

Proposition 4.5. *For $j \geq l$ the identity maps $\mathrm{id}: P(s)\to P(s)$ for all $s\leq l$ define a chain map*

$$\mathrm{Id}_{(l)}^{(j)}: \qquad P_\bullet(j) \to P_\bullet(l)[j-l]\langle j-l\rangle$$

which induces a non-trivial element in $\mathrm{Ext}^{j-l}(M(j),M(l))$. For $j>l$, the maps $f_{s,s-1}: P(s)\to P(s-1)$ for all $s\leq l+1$ define a chain map

$$F_{(l)}^{(j)}: \qquad P_\bullet(j) \to P_\bullet(l)[j-l-1]\langle j-l-2\rangle$$

which induces a non-trivial element in $\mathrm{Ext}^{j-l-1}(M(j),M(l))$.

Proof. We have to check that the maps are not nullhomotopic which is clear in the clear in the first case. For $F_{(l)}^{(j)}$, a homotopy would be a map $H \in \mathrm{Hom}^{j-l-2}(P_\bullet(j), P_\bullet(l)\langle j-l-2\rangle)$ which cannot exist by Lemma 3.9 since $j \not\leq l+1^2+(j-l-2)$. □

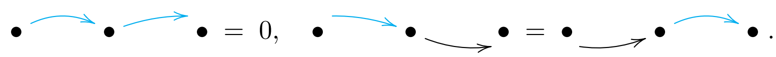
Theorem 4.1 implies that we constructed a basis of E_N^1. By composing chain maps we obtain the following relations in $\mathrm{Hom}(P_\bullet, P_\bullet)$:

$\mathrm{Id}_{(l)}^{(j)}\cdot \mathrm{Id}_{(m)}^{(l)} = \mathrm{Id}_{(m)}^{(j)}$, $F_{(l)}^{(j)}\cdot F_{(m)}^{(l)} = 0$, $\mathrm{Id}_{(l)}^{(j)}\cdot F_{(m)}^{(l)} = F_{(m)}^{(j)}$, $F_{(l)}^{(j)}\cdot \mathrm{Id}_{(m)}^{(l)} = F_{(m)}^{(j)}$

Reformulating the above result in terms of quivers, we obtain:

Theorem 4.6. *The algebra E_N^1 is isomorphic to the path algebra of the quiver*

with relations

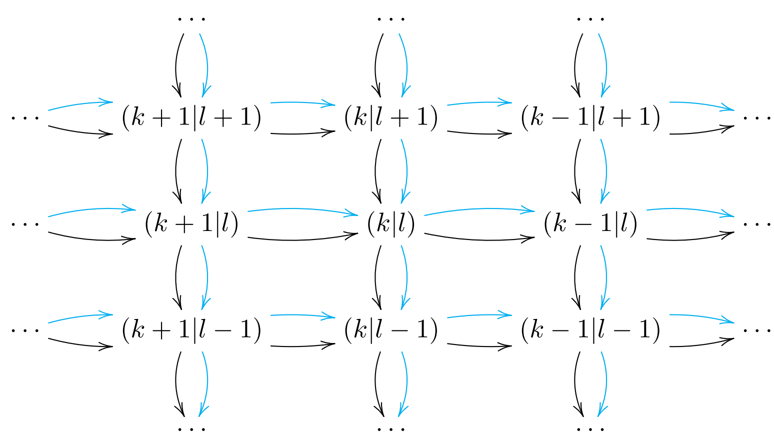


The vertex • labeled i corresponds to the idempotent e_λ where $\lambda = \lambda_0.s_1 \cdot \ldots s_i$.

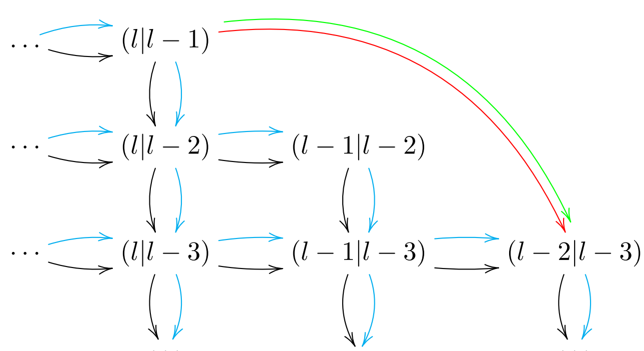
4.2. The result for $n = 2$

Now consider $(n, m) = (2, N-1)$. The elements in $W^{\mathfrak{p}}$ are precisely the elements $s_2 \cdot \ldots s_k \cdot s_1 \cdot \cdots \cdot s_l$ with $0 \leq l < k \leq N$. We denote the weight $\lambda = \lambda_0.s_2 \cdot \ldots \cdot s_k \cdot s_1 \cdot \ldots \cdot s_l$ by $(k|l)$; the associated diagrammatical weight has $\wedge$'s at the lth and kth position (starting to count with position zero).

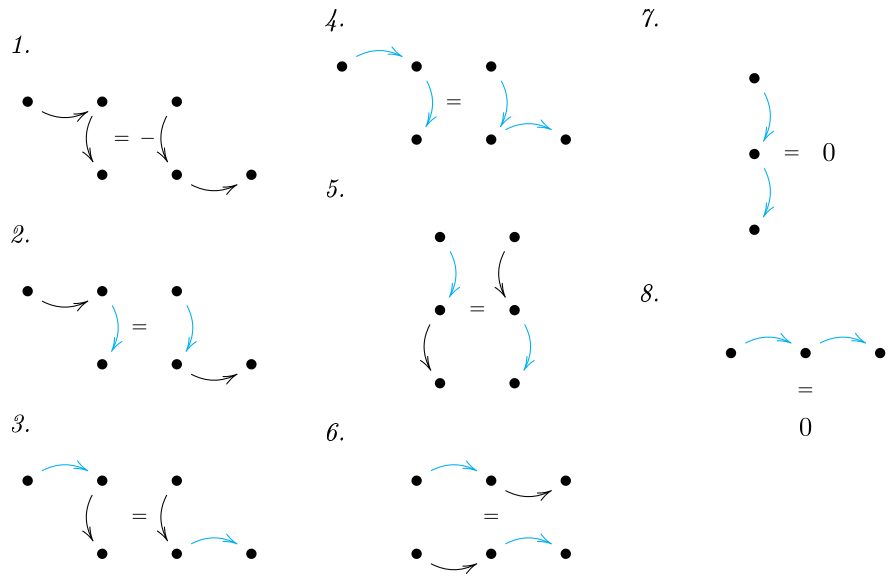
Theorem 4.7. *The algebra E_N^2 is isomorphic to the path algebra of the quiver*



for $k > l + 2$ and in the other cases:



with relations as follows (in case that both sides of the relation exist):



These are all relations for the middle part of the quiver, i.e. in the upper diagram. The additional relations for the boundaries can be found in [Kl].

5. The A_∞-structure on E^n_m

A_∞-algebras are a generalization of associative algebras, see [Ke] for an overview, including historical and topological motivation. A very detailed exposition with most of the proofs is provided in [L-H].

Definition 5.1. An A_∞-*algebra* over a field k is a $\mathbb{Z}$-graded k-vector space $A = \bigoplus_{p\in\mathbb{Z}} A^p$ endowed with a family of graded k-linear maps

$$m_n : A^{\otimes n} \to A,\ \ n \geq 1$$

of degree $2-n$ satisfying the following Stasheff identities:

$$\sum(-1)^{r+st} m_{r+t+1}(\mathrm{Id}^{\otimes r} \otimes m_s \otimes \mathrm{Id}^{\otimes t}) = 0$$

where for fixed n the sum runs over all decompositions $n = r+s+t$ with $s \geq 1$, and $r, t \geq 0$.

We use the *Koszul sign convention* $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$, for tensor products, where $x,\ y,\ f,\ g$ are homogeneous elements of degree $|x|, |y|, |f|, |g|$ respectively.

Definition 5.2. Let A and B be two A_∞-algebras. A *morphism of A_∞-algebras* $f : A \to B$ is a family $f_n : A^{\otimes n} \to B$ of graded k-linear maps of degree $1-n$ such that

$$\sum (-1)^{r+st} f_{r+t+1}(\mathrm{Id}^{\otimes r} \otimes m_s \otimes \mathrm{Id}^{\otimes t}) = \sum (-1)^w m_q(f_{i_1} \otimes \cdots \otimes f_{i_q})$$

for all $n \geq 1$. Here, the sum run over all decompositions $n = r+s+t$ and over all decompositions $n = i_1 + \cdots + i_q$ with $1 \leq q \leq n$ and all $i_s \geq 1$ respectively. The sign on the right-hand side is given by $w = \sum_{j=1}^{q-1}(q-j)(i_j-1)$.
A morphism f is a *quasi-isomorphism* if f_1 is a quasi-isomorphism. It is *strict* if $f_i = 0$ for all $i \neq 1$.

Our goal is to put an A_∞-structure on the Ext-algebras E_m^n. The first step is to introduce an A_∞-structure on the cohomology of an A_∞-algebra (the so-called minimal model) and then realize our Ext-algebra as the cohomology of an A_∞-algebra, namely the Hom-algebra introduced earlier.

Theorem 5.3 ([Ka1])**.** *Let A be an A_∞-algebra and $H^*(A)$ its cohomology. Then there is an A_∞-structure on $H^*(A)$ such that $m_1 = 0$ and m_2 is induced by the multiplication on A, and there is a quasi-isomorphism of A_∞-algebras $H^*(A) \to A$ lifting the identity of $H^*(A)$. Moreover, this structure is unique up to isomorphism of A_∞-algebras.*

All known (at least to us) proofs inductively construct the model, but the approaches are slightly different. We follow here Merkulov's more general construction [Me] in the special situation of a differential graded algebra:

Proposition 5.4 ([Me])**.** *Take (A, d) a differential graded algebra with grading shift $[\ \ ]$. Let $B \subset A$ be a vector subspace of A and $\Pi : A \to B$ a projection commuting with d. Assume that we are given a homotopy $Q : A \to A[-1]$ such that*

$$1 - \Pi \quad = \quad dQ + Qd. \tag{5.1}$$

Define $\lambda_n : A^{\otimes n} \to A$ for $n \geq 2$ by $\lambda_2(a_1, a_2) := a_1 \cdot a_2$ and recursively,

$$\begin{aligned} &\lambda_n(a_1, \ldots, a_n) \\ &= -\sum_{\substack{k+l=n \\ k,l \geq 1}} (-1)^{k+(l-1)(|a_1|+\cdots+|a_k|)} Q(\lambda_k(a_1, \ldots, a_k)) \cdot Q(\lambda_l(a_{k+1}, \ldots, a_n)). \end{aligned}$$

for $n \geq 3$, setting formally $Q\lambda_1 = -\mathrm{Id}$. Then the maps $m_1 = d$ and $m_n = \Pi(\lambda_n)$ define an A_∞-structure for a minimal model on B.

Choosing Q in a clever way simplifies computations, but our result will depend on this choice. We make our choices following [LPWZ]. To define Q, we first divide the degree n part A^n of A into three subspaces, for this, denote by Z^n the cocycles of A and by B^n the coboundaries. As we work over a field, we can find subspaces H^n and L^n such that

$$Z^n = B^n \oplus H^n \quad \text{and} \quad A^n = B^n \oplus H^n \oplus L^n. \tag{5.2}$$

We identify the nth cohomology group $H^n(A)$ via (5.2) with H^n and want to apply Proposition 5.4 with the choice $B = H^*(A)$, the projection Π to the direct summand H^* and the map Q defined as follows:

1. When restricted to Z^n by equation (5.1) and the condition that $d|_{Z^n}$ equals to zero, the map Q has to satisfy the relation $1 - \Pi = dQ$. In particular, $dQ|_H$ has to be zero. We choose $Q|_H = 0$.

2. On B^n the map Π is zero, and therefore the map $Q|_B$ has to satisfy $1 = dQ$, i.e. Q has to be a preimage of d. We want to choose this preimage as small as possible i.e. with no non-trivial terms from Z^n (they would anyway be annihilated by d). Since d is injective on L, we can choose $Q|_B = (d|_L)^{-1}$.

3. We briefly outline how to determine Q restricted to L (although it won't play any role in our computations later on). From (5.1) we get the restriction $1 = Qd + dQ$. As $d(a) \in B$ for all $a \in A$ we see that $Qd|_L = (d|_L)^{-1}d|_L = 1$, so we can define $Q|_L = 0$.

Now the construction of a minimal model applies to our situation if we choose $A := A_m^n := \mathrm{Hom}(P_\bullet, P_\bullet)$, where $P_\bullet$ is the direct sum of all linear projective resolutions of $M(\lambda)$, $\lambda \in \Lambda_m^n$ from 3.7, and $E = \mathrm{Ext}_m^n = H^*(A)$.

In the following we give an upper bound for the l with $m_l \neq 0$. Already in the case $n = 2$ we can show that not all m_l for $l > 2$ vanish and therefore our specific model provides interesting examples of A_∞-algebras with non-trivial higher multiplications. We start by stating the following Lemma generalizing the fact that the multiplication of two morphisms can only be non zero if they lie in appropriate Hom-spaces.

Lemma 5.5. *Let a_i, $1 \leq i \leq l$ be homogeneous elements of degree k_i in E_m^n of the form $a_i \in \mathrm{Ext}^{k_i}(M(\mu_i), M(\nu_i))$ $1 \leq i \leq l$. Then we have $\lambda_l(a_1, ..., a_l) = 0$ unless $\nu_i = \mu_{i+1}$ for all $1 \leq i \leq l-1$; and if $\lambda_l(a_1, ..., a_l) \neq 0$ we have $\lambda_l(a_1, ..., a_l) \in \mathrm{Hom}^{\Sigma k_i + 2 - l}(P_\bullet(\mu_1), P_\bullet(\nu_l))$.*

Proof. The proof goes by induction on l, using Theorem 5.4, see [Kl]. □

We obtain the following *General Vanishing Theorem*:

Theorem 5.6. *The A_∞-structure on E_m^n satisfies $m_l = 0$ for all $l > n^2 + 2$.*

Proof. We claim that $\lambda_l = 0$ if $l > n^2 + 2$. Since λ_l is linear, we can work with nonzero homogeneous basis elements and therefore by Lemma 5.5 take $a_i \in \mathrm{Ext}^{k_i}(M(\mu_i), M(\mu_{i+1}))$ for $1 \leq i \leq l$. By Lemma 4.2 there are $d_i \geq 0$ such that $k_i = l(\mu_i) - l(\mu_{i+1}) - d_i$ and therefore $\sum_{i=1}^{l} k_i = l(\mu_1) - l(\mu_{l+1}) - \sum_{i=1}^{l} d_i$. From Lemma 5.5 we know that $\lambda_l(a_1, ..., a_l) \in \mathrm{Hom}^{\Sigma k_i + 2 - l}(P_\bullet(\mu_1), P_\bullet(\nu_l))$. If $\lambda_l \neq 0$, then Lemma 3.9 implies $l(\mu_1) \leq l(\mu_{l+1}) + n^2 + \sum k_i + 2 - l$, thus

$$l(\mu_1) \quad \leq \quad l(\mu_{l+1}) + l(\mu_1) - l(\mu_{l+1}) - \sum_{i=1}^{l} d_i + 2 - l + n^2,$$

which is equivalent to $\sum_{i=1}^{l} d_i \leq n^2 + 2 - l$. Since $\sum_{i=1}^{l} d_i \geq 0$, we get $0 \leq n^2 + 2 - l$, equivalently $l \leq n^2 + 2$; providing the asserted upper bound. □

5.1. Explicit results for E_N^1 and E_{N-1}^2

In the previous section we established general vanishing results for the higher multiplications; in this section we describe explicit models for our small examples $n = 1$ and $n = 2$. The first result in this situation is the following:

Theorem 5.7 (1st vanishing Theorem)**.** *The algebra E_1^N is formal, i.e. there is a minimal model such that $m_n = 0$ for all $n \geq 3$.*

Proof. Recall that all multiplication rules in the algebra E_N^1 are already determined in $A_N^1 = \mathrm{Hom}(P_\bullet, P_\bullet)$. For all elements $a_1, a_2 \in \mathrm{Ext}(\oplus M(\lambda), \oplus M(\lambda)) = H^*(\mathrm{Hom}(P_\bullet, P_\bullet))$ identified with the subspace H^* via the decomposition from (5.2), the product $a_1 \cdot a_2$ also lies in the subspace H^* and has no boundary component in B^*. Since we have chosen $Q|_H = 0$, we obtain $Q(a_1 \cdot a_2) = 0$. Using the construction of the higher multiplications in Proposition 5.4 one gets $m_n = 0$ for all $n \geq 3$. □

The case of $n = 2$ turns out to be more interesting than the case $n = 1$ studied before, since we have non-vanishing higher multiplications. In contrast to the previous example this phenomenon is possible, since some multiplications in $A_{N-2}^1 = \mathrm{Hom}(P_\bullet, P_\bullet)$ are only homotopic to their product in the Ext-algebra. This yields the following theorem:

Theorem 5.8. *In the minimal model above, there are non-vanishing m_3.*

A complete list of all the higher multiplications m_3 is given in [Kl] where a detailed knowledge about the structure of projective resolutions us used to provides a stronger vanishing result than in the general case:

Theorem 5.9 (2nd Vanishing Theorem)**.** *The A_∞-structure on E_{N-2}^2 given by the construction above satisfies*

$$m_n = 0 \ \forall n \geq 4.$$

5.2. Ideas how to prove non-formality

The non-vanishing of higher multiplications established above does not answer the question whether the algebra is formal. To show that the algebra is not formal, we have to prove that no model exists such that $m_n = 0$ for all $n \geq 3$. As a tool one could use Hochschild cohomology. Given a dg-Algebra A one can compute its Hochschild cohomology by using the A_∞-structure on a minimal model of A (cf. [L-H, Lemma B.4.1] and [Ka2]). Assume that we have found a minimal model on $H^*(A)$ with $m_n = 0$ for $3 \leq n \leq p-1$. Then the multiplication m_p defines a cocycle for the Hochschild cohomology of A by the construction in [L-H, Lemma B.4.1]. If we can prove that this class is not trivial, we are done and have shown that the algebra is not formal. If we cannot, we have to modify our model such that $m_p = 0$ and then analyze whether m_{p+1} vanishes. This would go beyond the scope of this article.

Conjecture 5.10. *In general, the algebra E_m^n is not formal.*

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