Problem 1  Let \( \mathfrak{h} \) be a Lie subalgebra of a complex Lie algebra \( \mathfrak{g} \) such that \( \text{ad}_\mathfrak{g} h \) is semisimple for all \( h \in \mathfrak{h} \). Show that \( \mathfrak{h} \) is abelian. (4 points)

Remark: This shows that one condition in our definition of a Cartan subalgebra is automatic.

Problem 2  Show that up to isomorphism there are four rank two root systems. Draw pictures of these root systems. (4 points)

Problem 3  Let \( \mathfrak{g} \) be a Lie algebra with a decomposition

\[
\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_{\alpha}
\]

as a vector space where \( \mathfrak{h} \) is a commutative Lie subalgebra of \( \mathfrak{g} \), \( R \subset \mathfrak{h}^* \setminus \{0\} \) is a subset and \([h, x] = \alpha(h)x \) for all \( h \in \mathfrak{h} \) and \( x \in \mathfrak{g}_{\alpha} \). Show or disprove: then \( \mathfrak{g} \) is semisimple and \( \mathfrak{h} \) is a Cartan subalgebra. (4 points)

Problem 4  Let \( \mathfrak{h} \subset \mathfrak{g} = \mathfrak{so}(4, \mathbb{C}) \) be the subalgebra consisting of matrices of the form

\[
\begin{bmatrix}
0 & a & 0 & 0 \\
-a & 0 & 0 & 0 \\
0 & 0 & 0 & b \\
0 & 0 & -b & 0
\end{bmatrix}
\]

for \( a, b \in \mathbb{C} \). Show that \( \mathfrak{h} \) is a Cartan subalgebra of the semisimple Lie algebra \( \mathfrak{g} \) and find the corresponding root space decomposition. (4 points)