Let $k$ be a field. We abbreviate $\otimes = \otimes_k$ and $\text{Hom} = \text{Hom}_k$. All Lie algebras and algebras are defined over $k$. Algebras are associative with unit and homomorphisms of algebras respect the units.

**Problem 1**

(a) If $\mathfrak{g}$ is a semisimple complex Lie algebra show:

(i) Every finite dimensional $U(\mathfrak{g})$-module is semisimple. (1 point)

(ii) Not every $U(\mathfrak{g})$-module is semisimple. (1 point)

(b) Find a Lie algebra $\mathfrak{g}$ and a finite dimensional representation $V$ of $\mathfrak{g}$ that is not semisimple. (2 points)

**Problem 2**

In $U(\mathfrak{sl}(2, k))$ express the elements $ehf$, $e^2h^2f^2$, $hf^3$, $h^3f$ in the standard PBW basis $(f^lm^m e^n)_{l,m,n \in \mathbb{N}}$. (4 points)

**Problem 3**

Let $\mathfrak{g}$ be a Lie algebra over $k$. Show:

(a) There are unique homomorphisms of algebras

$$\Delta: U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$$

$$\eta: U(\mathfrak{g}) \to k$$

$$S: U(\mathfrak{g}) \to U(\mathfrak{g})^{\text{op}}$$

such that $\Delta(X) = X \otimes 1 + 1 \otimes X$ and $\eta(X) = 0$ and $S(X) = -X$ for all $X \in \mathfrak{g} \subset U(\mathfrak{g})$. (2 points)

(b) Let $\varepsilon: k \to U(\mathfrak{g})$, $\lambda \mapsto \lambda 1$, and $\mu: U(\mathfrak{g}) \otimes U(\mathfrak{g})$, $a \otimes b \mapsto ab$ be unit and multiplication of the algebra structure of $U(\mathfrak{g})$. Show that $(U(\mathfrak{g}), \mu, \varepsilon, \Delta, \eta, S)$ is a Hopf algebra over $k$. Is it commutative, cocommutative, involutive? Look up the relevant definitions if necessary. (2 points)

**Problem 4**

(a) If $(A, \mu, \varepsilon, \Delta, \eta, S)$ is a Hopf algebra over $k$ and $V, W$ are $A$-modules, show that $V \otimes W$ and $V^* = \text{Hom}(V, k)$ are $A$-modules in a natural way. (2 points)

(b) Let $\mathfrak{g}$ be a Lie algebra over $k$. Recall the following two statements from the lecture:

(i) If $V$ and $W$ are representations of $\mathfrak{g}$ so are $V^*$ and $V \otimes W$.

(ii) There is a canonical bijection between the class of representations of $\mathfrak{g}$ and the class of modules over $U(\mathfrak{g})$.

Consider $U(\mathfrak{g})$ as a Hopf algebra as in the previous problem. Show that the bijection from (ii) is compatible with taking duals and tensor products à la (i) and (a), respectively. (2 points)