Representation Theory I
Bonn, summer semester 2015

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Sheet 2

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Problem 1 Let V be a finite dimensional complex vector space. If a and b are commuting diagonalizable linear endomorphisms of V, then a and b are simultaneously diagonalizable. Deduce that $a + b$ is diagonalizable.

Does this generalize to any set of pairwise commuting endomorphisms of V? (4 points)

Problem 2 (Invariance Lemma) Let V be a finite-dimensional representation of a complex Lie algebra $\mathfrak{g}$ and let $I \subset \mathfrak{g}$ be an ideal. Let $\lambda \in I^*$ be a weight. Then

$$V_\lambda = \{v \in V \mid xv = \lambda(x)v \text{ for all } x \in I\}$$

(the λ-weight space of V considered as a representation of I) is a subrepresentation of $\mathfrak{g}$, i. e. $\mathfrak{g}(V_\lambda) \subset V_\lambda$. (4 points)

Hints: Assume that $V_\lambda \neq 0$.

(a) Reduce to the claim: $\lambda([x, y]) = 0$ for all $x \in I$ and $y \in \mathfrak{g}$.

(b) To see the claim, fix $y \in \mathfrak{g}$ and $0 \neq w \in V_\lambda$. Let $n \geq 0$ be maximal such that $w, yw, y^2w, \ldots, y^n w$ are linearly independent. Let W_i be the $\mathbb{C}$-span of $w, \ldots, y^i w$ (for $0 \leq i \leq n$). Then W_n is y-stable, and all W_i are z-stable where $z \in I$ is arbitrary. More precisely, the matrix of $z|_W : W_n \to W_n$ with respect to the basis $w, yw, \ldots, y^n w$ is upper triangular with diagonal entries $\lambda(z)$. Compute its trace. Now consider $z = [x, y]$.

Problem 3

(a) Let $\mathcal{N} \subset \mathfrak{sl}(2, \mathbb{C})$ denote the subset of nilpotent elements. Find a polynomial function f on $\mathfrak{sl}(2, \mathbb{C})$ such that $\mathcal{N}$ is the vanishing locus of f, i. e.

$$\mathcal{N} = \{x \in \mathfrak{sl}(2, \mathbb{C}) \mid f(x) = 0\}.$$ (1 point)

(b) Show that there is a natural bijection between the set of conjugacy classes of nilpotent elements in $\mathfrak{gl}(n, \mathbb{C}) = \mathrm{End}_\mathbb{C}(\mathbb{C}^n)$ and the set of partitions of n. How many such conjugacy classes are there for $n = 5$? (1 point)

(c) Consider the natural action of S_n on $\mathbb{C}^n$ by permutation and let $\mathbb{C}^n/S_n$ denote the set of S_n-orbits in $\mathbb{C}^n$. Show that mapping $A \in \mathfrak{gl}(n, \mathbb{C})$ to its set of eigenvalues with multiplicities defines a map $\mathfrak{gl}(n, \mathbb{C}) \to \mathbb{C}^n/S_n$. Show that this map induces a bijection between the set of conjugacy classes of diagonalizable elements in $\mathfrak{gl}(n, \mathbb{C})$ and $\mathbb{C}^n/S_n$. (1 point)

(d) What happens in (b) and (c) if $\mathfrak{gl}(n, \mathbb{C})$ is replaced by $\mathfrak{sl}(n, \mathbb{C})$? (1 point)

Problem 4 A Borel subalgebra of a Lie algebra is a maximal solvable Lie subalgebra. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$.

(a) Show that the set $\mathfrak{b}$ of upper triangular matrices in $\mathfrak{g}$ forms a Borel subalgebra. (1 point)

(b) Given any Borel subalgebra $\mathfrak{b}'$ of $\mathfrak{g}$, there is an element $g \in \mathrm{SL}(2, \mathbb{C})$ such that $g\mathfrak{b}'g^{-1} = \mathfrak{b}$, for $\mathfrak{b}$ as in (a). (2 points)

(c) Let $\mathfrak{b}$ be any Borel subalgebra of $\mathfrak{g}$. Consider the natural representation of $\mathfrak{b}$ on $\mathbb{C}^2$. Show that there is a unique one-dimensional subrepresentation $U_\mathfrak{b}$ of $\mathbb{C}^2$ and that the assignement $\mathfrak{b} \mapsto U_\mathfrak{b}$ defines a bijection

$$\{\text{Borels in } \mathfrak{g}\} \xrightarrow{\sim} \mathbb{P}^1\mathbb{C}.$$ (1 point)