Representation Theory I
Bonn, summer semester 2015

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# Sheet 1

Due Wednesday, April 22, 2015

**Problem 1** Determine the dimensions of $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{so}(2n+1, \mathbb{C})$, $\mathfrak{sp}(2n, \mathbb{C})$, $\mathfrak{so}(2n, \mathbb{C})$. (4 points)

**Problem 2** Show: Lie subalgebras and quotients of nilpotent Lie algebras are nilpotent. (4 points)

**Problem 3** Let $A = (a_{ij})_{i,j=1}^n$ be the Cartan Matrix of type $A_n$, for $n \leq 1$ and let $\mathfrak{g}$ be the complex Lie algebra generated by the $3n$ elements $\{e_i, f_i, h_i \mid 1 \leq i \leq n\}$ subject to the relations

$$[h_i, h_j] = 0 \qquad \text{(R1)}$$
$$[e_i, f_j] = \delta_{ij} h_i \qquad \text{(R2)}$$
$$[h_i, e_j] = a_{ji} e_j \qquad \text{(R3)}$$
$$[h_i, f_j] = - a_{ji} f_j$$
$$(\operatorname{ad} e_i)^{-a_{ji}+1}(e_j) = 0 \qquad (i \neq j) \qquad \text{(S1)}$$
$$(\operatorname{ad} f_i)^{-a_{ji}+1}(f_j) = 0 \qquad (i \neq j) \qquad \text{(S2)}$$

(a) Show that there is a surjective homomorphism $\varphi\colon \mathfrak{g} \to \mathfrak{sl}_{n+1}$ of Lie algebras. (2 points)

(b) Show for $n = 1$ that $\varphi$ is an isomorphism. (2 points)

(c) Bonus problem: Show for arbitrary $n \geq 1$ that $\varphi$ is an isomorphism. (2 points)

**Problem 4**

(a) Show that $\mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{sp}(2, \mathbb{C})$. (1/2 point)

(b) Show that $\mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{so}(3, \mathbb{C})$. (3/2 points)

(c) Show that $\mathfrak{sl}(4, \mathbb{C}) \cong \mathfrak{so}(6, \mathbb{C})$. (2 points)

Hint: Let $V$ be a 4-dimensional vector space. Then $W = \bigwedge^2 V$ has dimension six, and $W \times W \to \bigwedge^4 V$, $(x, y) \mapsto x \wedge y$, defines a symmetric non-degenerate bilinear form $\beta$ if we choose an isomorphism $\bigwedge^4 V \cong \mathbb{C}$. Then the Lie subalgebra $\mathfrak{g}_\beta$ of $\mathfrak{gl}(W)$ (see tutorial problem 2) is isomorphic to $\mathfrak{so}(6, \mathbb{C})$. On the other hand,

$$\rho\colon \mathfrak{sl}(V) \to \mathfrak{gl}(W),$$
$$X \mapsto (\rho(X)\colon v_1 \wedge v_2 \mapsto Xv_1 \wedge v_2 + v_1 \wedge Xv_2),$$

defines a homomorphism of Lie algebras.

These are examples of isomorphisms between classical simple Lie algebras. They show that the Lie algebras of type $A_1$ and type $C_1$ (resp. type $A_1$ and type $B_1$ resp. type $A_3$ and type $D_3$) are isomorphic; symbolically $A_1 = B_1 = C_1$ and $A_3 = D_3$. There are also isomorphisms $\mathfrak{sp}(4) \cong \mathfrak{so}(5)$ and $\mathfrak{sl}(2) \times \mathfrak{sl}(2) \cong \mathfrak{so}(4)$, symbolically $C_2 = B_2$ and $A_1 \times A_1 = D_2$. These isomorphisms explain that the "Dynkin diagrams" $B_1, C_1, C_2, D_2, D_3$ do not appear in the classification of simple complex Lie algebras ($D_1$ does not appear because $\mathfrak{so}(2)$ is abelian and therefore not semisimple by definition).