Set theory and model-theoretic logics

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One of the roles of logic is to serve as a tool for the study of structures. This role materializes in first order logic in the most spectacular way, as shown for example by the so-called Main Gap Theorem of Shelah. Typical of first order logic is that it cannot distinguish between cardinalities of infinite models. By Lindström's Theorem this property actually characterizes first order logic. There are many extensions of first order logic where sharper distinctions are possible. The most notable ones are the infinitary logics, logics with generalized quantifiers, and higher order logics. Infinitary logics are related to generalized recursion theory and set theory. Generalized quantifiers are related to model theory. Higher order logics are related to what could be called definability theory (following J. Addison).

There are also intermediate logics which do not fit well into the above three categories. A good example is the equicardinality quantifier "there are as many x with $\phi(x)$ as there are y with $\psi(y)$", which is a generalized quantifier and should therefore belong to the model theory category, but is actually an amalgam of model theory, set theory and definability theory. It is an example of a *strong* logic, that is, a logic which has enough power to express properties of not only this or that model, but of the underlying set theoretical universe. Other examples of strong logics are the Henkin quantifier, most higher order logics, and infinite quantifier logics $L_{\kappa\lambda}$, $\lambda > \omega$. The opposite is an *absolute* logic, that is, a logic the truth definition of which makes no reference to what kind of set there exists in the underlying universe. Typical examples of absolute logics are the infinitary logics $L_{\kappa\omega}$, enhanced perhaps with the game quantifier.

The first and foremost model theoretic properties of first order logic are the compactness property, the Löwenheim-Skolem property, the Craig interpolation property, and the axiomatization property. Each property has a life among extensions of first order logic, but often in a weaker form. By and large one expects that the logics in the "model theory" category, i.e. logics with generalized quantifiers, would permit such model theoretic properties as we just listed. It has turned out surprisingly difficult to prove such results. The truth is, the range of model theoretic properties extends all across the spectrum of logics, but in many cases only under subtle set theoretic assumptions.

In my work I try to get a clear picture of the set theoretic conditions on which model theoretic properties of extensions of first order logic depend. Note that the proofs of such properties do not always follow the line of argument familiar from first order logic. Genuinely new techniques have to be developed. Sometimes these techniques are or become standard tools in the model theory of first order logic (e.g. back-and-forth techniques, Vaughtian pairs). Sometimes they lead to set theoretic investigations completely detached from model theory (e.g. large cardinals, reflection principles for the continuum, the study of the ordering of trees).

In my recent work with Magidor we study a strong form of the Löwenheim-Skolem-Tarski property for the equicardinality quantifier and its relatives. To show that this property has great proof theoretic strength we prove that it implies failures of weak square and also that it implies the SCH. On the other hand we show, starting from a supercompact cardinal, that it can hold as low as on the first weakly inaccessible cardinal.