DIAMOND ON SUCCESSOR CARDINALS

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Abstract. We include a proof to the main result of Shelah’s paper 922, e.g., that for uncountable \( \lambda \), \( 2^{\lambda} = \lambda^+ \) iff \( \diamondsuit_{\lambda^+} \). The presentation follows a lecture given by Péter Komjáth at the HUJI seminar on 28/Dec/2007.

**Theorem** (Shelah). Suppose \( \lambda \) is a cardinal satisfying \( 2^{\lambda} = \lambda^+ \).

Then \( \diamondsuit_S \) holds for any stationary \( S \subseteq \{ \delta < \lambda^+ \mid \text{cf}(\delta) \neq \text{cf}(\lambda) \} \).

**Proof.** Fix a stationary set \( S \) as above. In particular, \( \lambda \) is uncountable. To avoid trivialities, we may also assume that \( S \cap \lambda = \emptyset \) and that \( S \) contains no successor ordinals. Set \( \kappa := \text{cf}(\lambda) \). For each \( \delta \in S \), let \( \{ A^\delta_i \mid i < \kappa \} \) be an increasing chain of elements of \( [\delta]^{<\lambda} \) satisfying \( \delta = \bigcup_{i<\kappa} A^\delta_i \).

For all \( \delta \in S \), since \( \text{cf}(\delta) < \lambda \), we may also assume that \( \sup(A^\delta_0) = \delta \).

**Notation.** For \( X \subseteq I \times Y \) and \( i \in I \), write \( (X)_i = \{ y \mid (i, y) \in X \} \).

**Lemma 1.** Suppose \( \{X_\beta \mid \beta < \lambda^+ \} \) is an enumeration of \( [\kappa \times (\lambda \times \lambda^+)]^{\leq \lambda} \).

Then there exists some \( i < \kappa \) such that for all \( Z \subseteq \lambda \times \lambda^+ \), the following is stationary:

\[
S_{i,Z} := \{ \delta \in S \mid \sup(\alpha \in A^\delta_0 \mid \exists \beta \in A^\delta_i (Z \cap (\lambda \times \alpha) = (X_\beta)_i)) = \delta \}.
\]

**Proof.** Suppose not. Then for all \( i < \kappa \), we may find some \( Z_i \subseteq \lambda \times \lambda^+ \) and a club \( D_i \subseteq \lambda^+ \) that avoids \( S_{i,Z} \). Define \( f : \lambda^+ \to \lambda^+ \) by:

\[
f(\alpha) := \min(\beta < \lambda^+ \mid X_\beta = \bigcup_{j<\kappa} (Z_j \cap (\lambda \times \alpha))) \]

Let \( D \subseteq \bigcap_{i<\kappa} D_i \) be a club such that \( f(\alpha) < \delta \) for all \( \alpha < \delta \in D \).

Clearly, for \( \delta \in D \):

\[
A^\delta_0 = \{ \alpha \in A^\delta_0 \mid \exists \beta \in A^\delta_i (Z_j \cap (\lambda \times \alpha) = (X_\beta)_j) \}.
\]

Fix \( \delta \in D \cap S \). For \( i < \kappa \), write:

\[
B^\delta_i := \{ \alpha \in A^\delta_0 \mid \exists \beta \in A^\delta_i \forall j < \kappa (Z_j \cap (\lambda \times \alpha) = (X_\beta)_j) \}.
\]

By \( A^\delta_0 = \bigcup_{i<\kappa} B^\delta_i \), sup \( A^\delta_0 = \delta \) and \( \text{cf}(\delta) \neq \kappa \), there must exist some \( i < \kappa \) with \( \sup(B^\delta_i) = \delta \). In particular:

\[
\sup(\alpha \in A^\delta_i \mid \exists \beta \in A^\delta_i (Z_j \cap (\lambda \times \alpha) = (X_\beta)_j)) = \delta
\]

i.e., \( \delta \in S_{i,Z} \). A contradiction to \( \delta \in D_i \).

\[\square\]

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Corollary 2. There exists a sequence \(\langle A^\delta \mid \delta \in [\delta]^<\lambda \rangle\), and an enumeration \(\{X_\beta \mid \beta < \lambda^+\} = [\lambda \times \lambda^+]^{\leq \lambda}\) such that for all \(Z \subseteq \lambda \times \lambda^+\), the following set is stationary:

\[
S_Z := \{ \delta \in S \mid \sup\{\alpha \in A^\delta \mid \exists \beta \in A^\delta (Z \cap (\lambda \times \alpha) = X_\beta)\} = \delta \}.
\]

Proof. Take \(i\) as above, and consider \(\langle A^\delta \mid \delta \in S\rangle\) and \(\{X_\beta \mid \beta < \lambda^+\}\). \(\square\)

Let us fix such sequence \(\langle A^\delta \mid \delta \in S\rangle\) and enumeration \(\{X_\beta \mid \beta < \lambda^+\}\).

We shall now recursively define a sequence of subsets of \(\lambda^+\), \(\{Y_\tau \mid \tau < \lambda\}\), and a \(\subseteq\)-decreasing sequence of clubs of \(\lambda^+\), \(\{E_\tau \mid \tau < \lambda\}\).

Notation. Whenever \(\langle Y_\tau \mid \tau < \gamma^+\rangle\) is defined, we shall denote for \(\delta \in S\):

\[
V_\gamma^\delta = \{ (\alpha, \beta) \in A^\delta \times A^\delta \mid \forall \tau < \gamma (Y_\tau \cap \alpha = (X_\beta)_\tau) \}.
\]

We start the recursion by letting \(E_0 = Y_0 = \lambda^+\). Suppose now \(\langle (Y_\tau, E_\tau) \mid \tau < \lambda\rangle\) has been defined for some \(\gamma < \lambda\). Clearly, for any set \(Y_\gamma\), and any \(\delta \in S\), we would have \(V_\gamma^\delta \supseteq V_{\gamma+1}^\delta\). If there exists a set \(Y_\gamma \subseteq \lambda^+\) and a \(\subseteq\)-decreasing sequence of clubs of \(\lambda^+\), \(\{E_\tau \mid \tau < \lambda\}\), then the recursion must terminate at some \(\gamma^* < \lambda\).

Claim 3. The recursion must terminate at some \(\gamma^* < \lambda\).

Proof. Suppose not, and let \(\langle Y_\tau \mid \tau < \lambda\rangle, \langle E_\tau \mid \tau < \lambda\rangle\) be the output sequences. Put \(E = \bigcap_{\tau < \lambda} E_\tau\) and \(Z = \bigcup_{\tau < \lambda} \{\tau\} \times Y_\tau\).

Fix \(\delta \in E \cap S_Z\). Then by definition of \(S_Z\):

\[
\sup\{\alpha \in A^\delta \mid \exists \beta \in A^\delta (Z \cap (\lambda \times \alpha) = X_\beta)\} = \delta
\]

In other words:

\[
\sup\{\alpha \in A^\delta \mid \exists \beta \in A^\delta, \forall \tau < \lambda (Y_\tau \cap \alpha = (X_\beta)_\tau)\} = \delta.
\]

It follows that \(\sup\{\alpha < \delta \mid \exists \beta < \delta, (\alpha, \beta) \in V_{\gamma}^\delta\} = \delta\) for all \(\gamma < \lambda\).

Since \(S_Z \subseteq S\), the recursive construction gives that \(\langle V_\gamma^\delta \mid \gamma < \lambda\rangle\) is a strictly \(\subseteq\)-decreasing sequence of subsets \(A^\delta \times A^\delta\), contradicting the fact that \(|A^\delta| < \lambda\). \(\square\)

Thus, let \(\gamma^*\) be the point at which the recursion terminates, and let \(\langle Y_\tau \mid \tau < \gamma^*\rangle, \langle E_\tau \mid \tau < \gamma^*\rangle\) be the resulted sequences. Set \(E = \bigcap_{\tau < \gamma^*} E_\tau\).

For every \(\delta \in S \cap E\), put:

\[
S_\delta := \bigcup\{(X_\beta)_{\gamma^*} \mid (\alpha, \beta) \in V_{\gamma^*}^\delta\}.
\]

Claim 4. \(\{S_\delta \mid \delta \in E \cap S\}\) exemplify \(\diamond_S\).
Proof. Assume towards a contradiction that there exists a set $Y \subseteq \lambda^+$ and a club $C \subseteq E$ such that $S_\delta \neq Y \cap \delta$ for all $\delta \in C \cap S$.

Following the notation of the recursion, write $Y_\tau := Y$.

Let $Z = \bigcup_{\tau \leq \gamma^*} \{ \tau \} \times Y_\tau$. Then, for $\delta \in C \cap S_Z$, we have:

$$\sup\{ \alpha \in A^\delta \mid \exists \beta \in A^\delta \forall \tau \leq \gamma^* (Y_\tau \cap \alpha = (X_\beta)_\tau) \} = \delta.$$ 

So, $\sup\{ \alpha < \delta \mid \exists \beta < \delta ((\alpha, \beta) \in V_\gamma^\delta) \} = \delta$, and also:

$$Y \cap \delta = \bigcup \{(X_\beta)_{\gamma^*} \mid (\alpha, \beta) \in V_\gamma^\delta \}. $$

It follows that if $V_\gamma^{\delta^*+1} = V_\gamma^\delta$, then $Y \cap \delta = S_\delta$. However, by the choice of $Y$ and $\delta \in C$, this is not the case, i.e., $V_\gamma^{\delta^*+1} \neq V_\gamma^\delta$.

But if $\sup\{ \alpha < \delta \mid \exists \beta < \delta ((\alpha, \beta) \in V_\gamma^\delta) \} = \delta$ and $V_\gamma^{\delta^*+1} \neq V_\gamma^\delta$ for all $\delta \in S \cap C$, this means that the recursion could have been continued using $Y$ and $C$, while it was terminated at $\gamma^*$. A contradiction. \hfill \Box

Remark. To see that the above theorem is optimal, we mention the following two results concerning successors of regular and singular cardinals.

Theorem (Shelah). $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ is consistent with the failure of $\diamondsuit_S$ for $S = \{ \alpha < \omega_2 \mid \text{cf}(\alpha) = \aleph_1 \}$.

Theorem (Magidor). Assume $\text{GCH}$ and that $\kappa$ is a measurable cardinal.

In the generic extension of prikry forcing, $\text{GCH}$ holds, $\kappa^+$ is a successor of a singular cardinal of countable cofinality, and $\diamondsuit_S$ fails for some stationary $S \subseteq \{ \alpha < \kappa^+ \mid \text{cf}(\alpha) = \aleph_0 \}$. 

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