Large cardinals and locally defined well–orders of the universe (with Sy Friedman)

David Asperó*

I am going to present a proof of the following theorem.

Theorem 0.1 *(GCH) There is a formula $\Phi(x, y)$ without parameters and there is a definable class–sized partial order $\mathcal{P}$ preserving ZFC, GCH and cofinalities, and such that*

(1) $\mathcal{P}$ forces that there is a well–order $\leq$ of the universe such that

$$\{(a, b) \in H(\kappa^+)^2 \,:\, \langle H(\kappa^+), \in\rangle \models \Phi(a, b)\}$$

is the restriction $\leq\restriction H(\kappa^+)^2$ and is a well–order of $H(\kappa^+)$ whenever $\kappa \geq \omega_2$ is a regular cardinal, and

(2) for all regular cardinals $\kappa \leq \lambda$, if κ is a λ–supercompact cardinal in V, then κ remains λ–supercompact after forcing with $\mathcal{P}$.

One key task (Task 1) in the proof of Theorem 0.1 is this: For a fixed regular cardinal $\kappa \geq \omega_2$, we build a forcing iteration for manipulating certain weak guessing properties for club–sequences defined on stationary subsets of κ, in such a way that (a certain definable subset of) the set of ordinals τ for which there is some club–sequence on κ of height τ and satisfying the property codes any prescribed subset A of κ.[1]

Another task (Task 2) is the following: For the same fixed κ, given a function $F : \kappa \longrightarrow \mathcal{P}(\kappa)$ and a sequence $\mathcal{S} = \langle S_i \,:\, i < \kappa\rangle$ of pairwise disjoint stationary subsets of κ, we force in such a way that every $B \subseteq \kappa$ gets coded by some ordinal in δ^+ with respect to F and $\mathcal{S}$. This means that there is a

*ICREA and U. Barcelona

[1]A club–sequence $\langle C_\alpha \,:\, \alpha \in dom(\vec{C})\rangle$ has height τ iff $ot(C_\alpha) = \tau$ for all $\alpha \in dom(\vec{C})$.

club $E \subseteq \mathcal{P}_\kappa(\delta)$ such that for every $X \in E$ and every $i < \kappa$, if $X \cap \kappa \in S_i$, then $ot(X) \in F(X \cap \kappa)$ if and only if $i \in B$.

It is possible to add F and $\mathcal{S}$ as above, then to pick a subset A of κ coding F and $\mathcal{S}$, and then to perform Tasks 1 and 2 simultaneously, for A and for F and $\mathcal{S}$, by a nicely behaved[2] forcing. This is the *one–step construction at* κ.

The forcing $\mathcal{P}$ can be roughly described as a two–step iteration $\mathcal{B} * \dot{\mathcal{C}}$ in which $\mathcal{B}$ is a forcing iteration of length Ord adding a system of bookkeeping functions and $\dot{\mathcal{C}}$ is another iteration on which we force with the one–step forcing at κ for all the relevant κ (using the bookkeeping functions added by $\mathcal{B}$).

I intend to present the above one–step construction with some detail and to outline the general lifting lemma that we use in the large cardinal preservation part of the proof.

[2] κ–strategically closed and κ–c.c.