Subtle and Ineffable Tree Properties

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It is well known that an inaccessible κ is Mahlo iff there exists no special κ-Aronszajn tree and that it is weakly compact iff there exists no κ-Aronszajn tree (which we will abbreviate by κ-TP). For $(T, <_T)$ a tree[1] let us define the *subtle tree property* STP and the *ineffable tree property* ITP:

(STP(T)) If $\mathrm{ht}(T) = \kappa$, $C \subset \kappa$ club, $\langle t_\alpha \mid \alpha \in C\rangle \in \prod_{\alpha\in C} T_\alpha$, then there are $\alpha, \beta \in C$ such that $t_\alpha <_T t_\beta$,

(ITP(T)) If $\mathrm{ht}(T) = \kappa$, $\langle t_\alpha \mid \alpha < \kappa\rangle \in \prod_{\alpha<\kappa} T_\alpha$, then there is a stationary $S \subset \kappa$ such that $\{t_\alpha \mid \alpha \in S\}$ is a $<_T$-chain.

Now it is obvious from the usual definitions that an inaccessible κ is subtle iff every κ-tree T satisfies $\mathsf{STP}(T)$, for which we shall just write κ-STP, iff the complete binary tree $2^{<\kappa}$ satisfies $\mathsf{STP}(2^{<\kappa})$, and similarly for ineffability (and one can take this as a definition if unfamiliar with the concepts).

By [Mit73] one can collapse a weakly compact (a Mahlo) cardinal onto ω_2 such that in the resulting universe there exists no ω_2-Aronszajn tree (no special ω_2-Aronszajn tree), and if there are no ω_2-Aronszajn trees (no special ω_2-Aronszajn trees), then (κ is weakly compact)^L ((κ is Mahlo)^L) holds. One can do the same for subtlety and ineffability, so that the existence of a subtle or an ineffable cardinal is also equiconsistent with the truth of certain combinatorial principles for ω_2.

In [MS96] it is shown that if λ is the singular limit of strongly compact cardinals, then λ^+-TP holds—what about λ^+-STP or λ^+-ITP? Furthermore the consistency of ω_ω^+-TP is proved under some large cardinal assumptions, so can we get ω_ω^+-STP or ω_ω^+-ITP here? Baumgartner showed PFA implies ω_2-TP (see [Tod84, chap. 7] or [Dev83, §5]), so we would like to know if PFA also implies ω_2-STP or ω_2-ITP.

We can further generalize these properties to get ideals similar to the approachability ideal. For example we can consider the ideal of all subsets B of κ such that some κ-tree has an antichain which has an element of height β for every $\beta \in B$, so that κ-STP becomes the property this is a proper ideal. One can also reduce the requirement of having a tree with an antichain to having an antichain where the initial segments are enumerated before, so that we get an ideal containing the approachability ideal. We are then led to the question if for example on ω_2 these ideals can be the nonstationary ideals on $\mathrm{cof}(\omega_1)$, cf. [Mit05].

References

[Dev83] K. J. Devlin, *The Yorkshireman's guide to proper forcing*, Surveys in set theory, London Math. Soc. Lecture Note Ser., vol. 87, Cambridge Univ. Press, Cambridge, 1983, pp. 60–115. MR 823776, Zbl 0524.03041

[1] We require all trees to not split at limit levels, i.e. if δ is limit and $s, t \in T_\delta$ are such that $\{u \in T \mid u <_T s\} = \{u \in T \mid u <_T t\}$, then $s = t$. Otherwise the following concepts would just trivially be wrong.

[Mit05] W. Mitchell, *$I[\omega_2]$ can be the nonstationary ideal on* $\mathrm{Cof}(\omega_1)$, submitted to the proceedings of the Midrasha Mathematicae: *Cardinal Arithmetic at Work*, held March 2004 at the Hebrew University, Jerusalem, 2005? arXiv:math.LO/0407225

[Mit73] W. Mitchell, *Aronszajn trees and the independence of the transfer property*, Ann. Math. Logic **5** (1972/73), 21–46. MR 313057, Zbl 0255.02069

[MS96] M. Magidor and S. Shelah, *The tree property at successors of singular cardinals*, Arch. Math. Logic **35** (1996), no. 5-6, 385–404. MR 1420265, Zbl 0874.03060, arXiv:math.LO/9501220

[Tod84] S. Todorčević, *Trees and linearly ordered sets*, Handbook of set-theoretic topology, North-Holland, Amsterdam, 1984, pp. 235–293. MR 776625, Zbl 0557.54021