Research statement, Brian Semmes

In my thesis, I introduce a two-player game $G(f)$ which "characterizes" the Borel functions on the Baire space, in the sense that Player II has a winning strategy in $G(f)$ if and only if f is Borel. In this game, there are two players who alternate moves for ω rounds. Player I plays natural numbers $x_i \in \omega$ and Player II plays functions $\phi_i : T_i \to {}^{<\omega}\omega$ such that $T_i \subset {}^{<\omega}\omega$ is a finite tree, ϕ_i is monotone and length-preserving, and $i < j \Rightarrow \phi_i \subseteq \phi_j$.

$$\begin{array}{lcccccccc} \text{I:} & x_0 & & x_1 & & x_2 & & & x = \langle x_0, x_1, \dots \rangle \\ & & & & & & & \cdots & \\ \text{II:} & & \phi_0 & & \phi_1 & & \phi_2 & & \phi = \bigcup \phi_i \end{array}$$

After infinitely many rounds, Player I produces x and Player II produces ϕ as shown. Player II wins the game if and only if $\mathrm{dom}(\phi)$ has a unique infinite branch z and

$$\bigcup_{s \subset z} \phi(s) = f(x).$$

One of the results of my thesis is that Player II can guarantee victory in this game precisely when f is Borel. This is a generalization of the Wadge game, which characterizes the continuous functions in a similar way.

By adding extra rules for Player II, it is possible to characterize subclasses of Borel functions. In particular, it is possible to characterize Baire class 1 and Baire class 2. Using game-theoretic methods, I proved decomposition theorems for two subclasses of Baire class 2 (see notation section for what is meant by $n{\to}m$):

A function $f : {}^{\omega}\omega \to {}^{\omega}\omega$ is $2{\to}3 \Leftrightarrow$ there is a $\boldsymbol{\Pi}^0_2$ partition $\langle A_n : n \in \omega \rangle$ of ${}^{\omega}\omega$ such that $f \restriction A_n$ is Baire class 1.

A function $f : {}^{\omega}\omega \to {}^{\omega}\omega$ is $3{\to}3 \Leftrightarrow$ there is a $\boldsymbol{\Pi}^0_2$ partition $\langle A_n : n \in \omega \rangle$ of ${}^{\omega}\omega$ such that $f \restriction A_n$ is continuous.

At least for the Baire space, this extends the result of Jayne and Rogers (1982) that f is $2{\to}2$ if and only if there is a closed partition A_n such that $f \restriction A_n$ is continuous.

Notation:
The symbol ω denotes the set of natural numbers,
${}^{<\omega}\omega$ is the set of finite sequences of natural numbers,
${}^{\omega}\omega$ is the set of infinite sequences of natural numbers,
$T \subseteq {}^{<\omega}\omega$ is a tree if $t \in T$ and $s \subset t \Rightarrow s \in T$,
$\phi : T \to {}^{<\omega}\omega$ is monotone if $s \subseteq t \Rightarrow \phi(s) \subseteq \phi(t)$,
ϕ is length-preserving if $\mathrm{lh}(\phi(s)) = \mathrm{lh}(s)$, and
f is $n{\to}m$ if $f^{-1}[X] \in \boldsymbol{\Sigma}^0_m$ for every $X \in \boldsymbol{\Sigma}^0_n$.