Research Statement

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My research area is forcing technics of adding new reals related to special MAD families and cardinal invariants of the continuum. My advisor, Lajos Soukup and I are working on the following paper right now:

Invariants of analytic P-ideals and related forcing problems. (only a possible title) An ideal $\mathcal{I}$ on ω is *analytic* if as a subset of the space $\mathcal{P}(\omega)$ with the usual topology (i.e. Cantor-set) is analytic; $\mathcal{I}$ is a *P-ideal* if for each countable $\{I_n : n \in \omega\} \subseteq \mathcal{I}$ there is an $I \in \mathcal{I}$ such that $I_n \subseteq^* I$ (i.e. $|I_n \setminus I| < \omega$) for each n. It is well-known that each analytic P-ideal is of the form $\mathrm{Exh}(\varphi) = \{X \subseteq \omega : \lim \varphi(X \setminus n) = 0\}$ where $\varphi : \mathcal{P}(\omega) \to [0, \infty)$ is a *finite lower semicontinuous submeasure*. The main examples of such ideals are *density* and *summable* ideals.

Density ideals: Let $\{P_k : k \in \omega\}$ be a partition of ω into pairwise disjoint finite sets and let $\vec{\mu}$ be a sequence $\langle \mu_k : k \in \omega \rangle$ of measures such that μ_k is concentrated on P_k and $\limsup \mu_k(P_k) > 0$. Let $\mathcal{Z}_{\vec{\mu}}$ be the following ideal on ω:

$$\mathcal{Z}_{\vec{\mu}} = \{X \subseteq \omega : \lim \mu_k(X \cap P_k) = 0\}.$$

Ideals of this form are called density ideals. The ideal of asymptotic density zero sets, $\mathcal{Z} = \{A \subseteq \omega : \lim \frac{|A \cap n|}{n} = 0\}$ is a density ideal.

Summable ideals: Let $h : \omega \to \mathbb{R}^+$ be a function with $\sum_{n\in\omega} h(n) = \infty$ and let $\mathcal{I}_h$ be the following ideal on ω:

$$\mathcal{I}_h = \Big\{A \subseteq \omega : \sum_{n \in A} h(n) < \omega\Big\}.$$

These ideals are called summable ideals. For example the ideal of finite sets is a summable ideal.

Let $\mathcal{I}$ be an ideal on ω, and let $\mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$. An infinite family $\mathcal{M} \subseteq \mathcal{I}^+$ is *$\mathcal{I}$-almost disjoint ($\mathcal{I}$-AD)* if $A \cap B \in \mathcal{I}$ for each distinct $A, B \in \mathcal{M}$. An $\mathcal{I}$-AD family $\mathcal{M}$ is *maximal ($\mathcal{I}$-MAD)* if for each $X \in \mathcal{I}^+$ there is an $A \in \mathcal{M}$ such that $X \cap A \in \mathcal{I}^+$, that is, $\mathcal{M}$ is $\subseteq$-maximal among $\mathcal{I}$-AD families. The *almost disjoint number of* $\mathcal{I}$, denoted by $\mathfrak{a}_{\mathcal{I}}$ ($\mathfrak{a}^*_{\mathcal{I}}$), is the minimum of the cardinalities of (uncountable) $\mathcal{I}$-MAD families. We have proved the following results:

$\mathfrak{a}_{\mathcal{I}_h} > \omega$ for each summable ideal $\mathcal{I}_h$ and $\mathfrak{a}_{\mathcal{Z}_{\vec{\mu}}} = \omega$ for most density ideals.

$\mathfrak{a}^*_{\mathcal{Z}_{\vec{\mu}}} \leq \mathfrak{a}$ for each density ideal $\mathcal{Z}_{\vec{\mu}}$, where $\mathfrak{a} = \mathfrak{a}_{[\omega]^{<\omega}}$ is the well-known almost disjointness number.

$\mathfrak{b} \leq \mathfrak{a}^*_{\mathcal{I}}$ for each analytic P-ideal, where $\mathfrak{b}$ is the unbounding number of $\langle \omega^\omega, \leq^* \rangle$.

We are working on related forcing questions as well. Let V be a transitive model of (a large enough segment of) ZFC. An $X \subseteq \omega$ is a *$\mathcal{Z}$-covering real over V* if $X \in \mathcal{Z}$ and $A \subseteq^* X$ for each $A \in \mathcal{Z} \cap V$. Results:

If $V \subseteq W$ are models and W contains a $\mathcal{Z}$-covering real over V then W contains a dominating real over V as well.

If $V \subseteq W$ are models and W contains a slalom over V, that is, an $S \in W$, $S : \omega \to [\omega]^{<\omega}$, $|S(n)| \leq n$, and for each $f \in \omega^\omega \cap V$ $f(n) \in S(n)$ for all but finite n, then W contains a $\mathcal{Z}$-covering real over V. Specially the Localization-forcing ($\mathbb{LOC}$) adds $\mathcal{Z}$-covering reals.

There is a natural ccc forcing too which adds a $\mathcal{Z}$-covering real over the ground model but a σ-centered forcing notion cannot add such a real.

A forcing notion $\mathbb{P}$ has the Sacks-property if, and only if $\mathbb{P}$ is *$\mathcal{Z}$-bounding*, that is, $\mathbb{P}$ forces that for each new element of $\mathcal{Z}$ can be covered by an element of $\mathcal{Z}$ from the ground model.