Alex Hellsten
University of Helsinki

Research statement

My research has concentrated on weakly compact cardinals and the ideal naturally associated with them. A subset E of a regular cardinal κ is called Π^1_1-*indescribable* or *weakly compact* if for every Π^1_1-sentence ϕ and every $U \subseteq \kappa$ such that $\langle V_\kappa, \in, U\rangle \models \phi$ there exists an ordinal $\alpha \in E$ such that $\langle V_\alpha, \in, U \cap \alpha\rangle \models \phi$. Thus κ is a weakly compact cardinal iff there exists a weakly compact subset of κ or equivalently if κ is weakly compact as a subset of itself.

The *weakly compact ideal* consists of those subsets of κ that are not weakly compact. It is a normal ideal. It seems that weak compactness of sets is a remarkably natural generalisation of stationarity. This is one of the motivational factors behind the research. The phenomenon is also easily seen to generalise to the ideals associated with Π^1_n-indescribable cardinals for $n < \omega$.

Subtlety and ineffability, various diamond principles and saturation properties of the ideals are investigated. The weakly compact diamond is just like the ordinary diamond, except that guessing happens on a wekly compact set rather than just a stationary set. Weakly compact diamond holds on an ineffable cardinal whereas the classical diamond restricted to the regulars is known to hold on a subtle cardinal. One question we try to look into is whether κ can be subtle even though the weakly compact diamond fails. Other "small large cardinals" such as strongly unfoldables are of interest too.

It is consistent relative to a measurable cardinal that the weakly compact ideal over κ is not κ^+-saturated. How this result can be generalised to Π^1_n even for $n = 2$ is still unsolved.

The weakly compact ideal over κ is nowhere κ-saturated. It is open whether this result holds for ordinal (or weak) Π^1_1-indescribability, the concept that arises when inaccessibility is dropped from weak compactness. Characterisations via elementary embeddings and Π^1_1-sentences work in this kind of setting too, but some of the known characterisations of weak compactness imply inaccessibility. Models with large cardinals but many weak inaccessibles that are not inaccessible seem to be rather obscure, but motivation to pursue research in this direction can come from the thought that the true combinatorics underlying weak compactness can be better understood if inaccessibility is not in the picture.

All forcing arguments used so far have involved iterations with Easton supports (reverse Easton). It seems difficult or even impossible to find a really useful general preservation theorem for weak compactness. An observation on a very intuitive level is that elementary embeddings tend to be useful in forcing arguments wheras they do not seem to work very well in arguments that stay in one particular model of set theory. New ideas may be needed for results such as the nowhere κ-saturation mentioned above.