LECTURE NOTES: INTRODUCTION TO MATHEMATICAL LOGIC

PHILIPP SCHLICHT

Contents

These are lecture notes in progress from the University of Bonn in the summer of 2021. The lemmas, theorems etc. and their numbering will remain fixed, but some corrections and details will be added during the semester. I would like to thank the participants of the course for helpful comments and questions.

Overview

In the first chapter, we study structures, formulas and introduce the Hilbert calculus.

In the second chapter, we give an introduction to set theory. We begin informally with ordinals and cardinals, and then study axiomatic set theory up to transfinite induction. This can be seen as a foundation on which all results in this course are built.

In the third chapter, we present the completeness of Hilbert's proof calculus. We then study the compactness theorem and applications, deriving finitary analogues of infinitary combinatorial statements such as the infinite Ramsey's theorem.

In the fourth chapter, the main goal is Gödel's first incompleteness theorem. It shows that no matter how one extends the theory of the natural numbers, assuming there is a reasonable listing of all axioms, some statements that can neither be proved nor disproved will always remain.

In the final chapter, we give an introduction to model theory. We aim for some applications in algebra, for instance the Lefschetz principle, which relates statements about the complex numbers to other algebraically closed fields.

1. Formal languages and structures

Lecture 1
12. April

Mathematical logic studies formal languages and proofs (syntax), structures such as groups, fields, graphs or linear orders, and the connection between languages and structures (semantics). Expressions in a formal language are themselves considered as mathematical objects. For instance, a word in a language is a finite sequence of symbols, i.e. a function.

Date: April 30, 2021.

1.1. **Structures and formulas.** We begin by introducing structures and formulas in *first-order logic*.[1] Many familiar mathematical structures consist of a set with additional structure, for example:

(a) A *graph* is a pair (G, E), where $G \neq \emptyset$ is the set of nodes and $E \subseteq G^2$ is the set of *edges*, a *symmetric* set of ordered pairs in G. (A subset E of G^2 is called symmetric if $\forall x, y \in G\ (x, y) \in E \leftrightarrow (y, x) \in E$.)
(b) A *partial order* is a pair $(P, \leq)$, where $P \neq \emptyset$ is a set and $\leq$ is a binary relation on P satisfying the following conditions:
   (i) (Reflexivity) $\forall x \in P\ x \leq x$
   (ii) (Antisymmetry) $\forall x \in P\ ((x \leq y \wedge y \leq x) \rightarrow x = y)$
   (iii) (Transitivity) $\forall x \in P\ ((x \leq y \wedge y \leq z) \rightarrow \leq z)$

In general, a structure is defined as follows:

Definition 1.1.1. A *structure* or *model* is a pair $\mathcal{M} = (M, \mathcal{F})$, where M is a nonempty set and $\mathcal{F} = \langle F_i \mid i \in I \rangle$ is a family of

(1) elements (constants) $F_i \in M$,
(2) functions $F_i \colon M^{k_i} \rightarrow M$ with $k_i \in \mathbb{N}$, and
(3) relations $F_i \subseteq M^{k_i}$ with $k_i \in \mathbb{N}$.

and I is a set. Note that in most cases, I will be finite or countable.

When we write $\mathcal{M}$ and $\mathcal{N}$, we will assume that $\mathcal{M} = (M, \mathcal{F})$ and $\mathcal{N} = (N, \mathcal{F})$ as above.

Here are some more examples. The superscript notation will be defined in Definition 1.1.5.

Example 1.1.2.

(1) A ring $(R, 0^R, 1^R, +^R, \cdot^R)$.
(2) A group $(G, 1^G, \cdot^G, (\cdot^{-1})^G)$.
(3) The structure of the natural numbers $(\mathbb{N}, 0^{\mathbb{N}}, S^{\mathbb{N}}, +^{\mathbb{N}}, \cdot^{\mathbb{N}}. <^{\mathbb{N}})$, where $S^{\mathbb{N}}$ denotes the successor function.
(4) The field $(\mathbb{Q}, 0^{\mathbb{Q}}, 1^{\mathbb{Q}}, +^{\mathbb{Q}}, \cdot^{\mathbb{Q}})$.

All groups have a binary operation (multiplication), a neutral element and an inverse function. This is encoded in the language of groups.

Definition 1.1.3. A *language* or *alphabet* is a set of constant symbols, function symbols and relation symbols. Function and relation symbol have an *arity*, i.e. a number of arguments, $k \in \mathbb{N}$ with $k \geq 1$. For example, an k-ary function on a set M is of the form $f \colon M^k \rightarrow M$. A k-ary relation R on a set M is of the form $R \subseteq M^k$.

Here are some examples of languages.

Example 1.1.4.

(1) The empty language $\mathcal{L}_\emptyset = \emptyset$.
(2) The language $\mathcal{L}_R = \{0, 1, +, \cdot\}$ of rings and fields.
(3) The language $\mathcal{L}_G = \{1, \cdot, {}^{-1}\}$ of groups.
(4) The language $\mathcal{L}_O = \{<\}$ of strict linear ordes.
(5) The language $\mathcal{L}_{OF} = \mathcal{L}_R \cup \mathcal{L}_O$ of linearly ordered fields.
(6) The language $\mathcal{L}_{\mathbb{N}} = \{0, S, +, <\}$ of the natural numbers.
(7) The language $\mathcal{L}_\in = \{\in\}$ of set theory.

Let always c, d denote constant symbols, f, g function symbols and R, S relation symbols.

[1] Second-order logic, which we won't study here, allows two kinds of objects, for instance natural numbers and sets of natural numbers.

Definition 1.1.5. Suppose that $\mathcal{L}$ is a language. An $\mathcal{L}$-structure is a structure $\mathcal{M} = (M, \mathcal{F})$, where M is a nonempty set and $\mathcal{F} = \langle s^M \mid s \in \mathcal{L}\rangle$ and

(1) $s^{\mathcal{M}} \in M$ if $c \in \mathcal{L}$ is a constant symbol,
(2) $f^{\mathcal{M}} \colon M^k \to M$ if $f \in \mathcal{L}$ is a k-ary function symbol, and
(3) $R^{\mathcal{M}} \subseteq M^k$ if $R \in \mathcal{L}$ is a k-ary relation symbol.

So every symbol has an interpretation as an element, function or relation in the structure.

For example, let $\mathcal{R} = (\mathbb{R}, 0^{\mathcal{R}}, 1^{\mathcal{R}}, +^{\mathcal{R}}, \cdot^{\mathcal{R}})$ denote the field of real numbers, a structure in the language $\mathcal{L}_R = \{0, 1, +, \cdot\}$ of rings. Here an otherwise, we will often confuse the structure with its underlying set and write $(\mathbb{R}, 0^{\mathbb{R}}, 1^{\mathbb{R}}, +^{\mathbb{R}}, \cdot^{\mathbb{R}})$. One can further simplify the notation to $(\mathbb{R}, 0, 1, +, \cdot)$ when it is clear that one means constants and functions rather than symbols.

The familiar notions of homomorphisms, embeddings and isomorphism of (e.g.) groups, vector spaces etc. make sense in this general setting:

Definition 1.1.6. Suppose that $\mathcal{M} = (M, \langle s^{\mathcal{M}} \mid s \in \mathcal{L}\rangle)$ and $\mathcal{N} = N, \langle s^{\mathcal{N}} \mid s \in \mathcal{L}\rangle)$ are $\mathcal{L}$-structures. By a function $h \colon \mathcal{M} \to \mathcal{N}$, we mean a function $h \colon M \to N$ on the underlying sets.

(1) h is a *homomorphism* if for all $n \in \mathbb{N}$ and all $a_0, \dots, a_{n-1} \in M$:
    (a) $h(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for all constant symbols c.
    (b) $h(f^{\mathcal{M}}(a_0, \cdots, a_{k-1})) = f^{\mathcal{N}}(h(a_0), \cdots, h(a_{k-1}))$ for all k-ary function symbols f.
    (c) $R^{\mathcal{M}}(a_0, \cdots, a_{k-1}) \implies R^{\mathcal{N}}(h(a_0), \cdots, h(a_{k-1}))$ for all k-ary relation symbols R.
(2) h is an *embedding* if it is an injective homomorphism and for all k-ary relation symbols R and $a_0, \dots, a_{k-1} \in M$,
$$R^{\mathcal{M}}(a_0, \cdots, a_{k-1}) \iff R^{\mathcal{N}}(h(a_0), \cdots, h(a_{k-1})).$$
(3) h is an *isomorphism* if it is a surjective embedding.
(4) h is an *automorphism* if it is an isomorphism and $\mathcal{M} = \mathcal{N}$.

The notion of subgroup, subfield etc. make sense in this general setting.

Definition 1.1.7. Suppose that $\mathcal{M} = (M, \langle s^{\mathcal{M}} \mid s \in \mathcal{L}\rangle)$ and $\mathcal{N} = N, \langle s^{\mathcal{N}} \mid s \in \mathcal{L}\rangle)$ are $\mathcal{L}$-structures.

(1) $\mathcal{M}$ is a *substructure* of $\mathcal{N}$ if $M \subseteq N$ and the identity $\mathrm{id} \colon \mathcal{M} \to \mathcal{N}$ is an embedding, i.e. for all $n \in \mathbb{N}$ and all $a_0, \dots, a_{n-1} \in M$:
    (a) $c^{\mathcal{M}} = c^{\mathcal{N}}$ for all constant symbols $c \in \mathcal{L}$.
    (b) $f^{\mathcal{M}}(a_0, \cdots, a_{k-1}) = f^{\mathcal{N}}(a_0, \cdots, a_{k-1})$ for all k-ary function symbols $f \in \mathcal{L}$.
    (c) $R^{\mathcal{M}}(a_0, \cdots, a_{k-1}) \iff R^{\mathcal{N}}(a_0, \cdots, a_{k-1})$ for all k-ary relation symbols $R \in \mathcal{L}$.
(2) $\mathcal{N}$ is a *superstructure* of $\mathcal{M}$ if $\mathcal{M}$ is a substructure of $\mathcal{N}$.

One can also change a structure by adding or removing constants, functions or relations.

Definition 1.1.8. Suppose that $\mathcal{K} \subseteq \mathcal{L}$ are languages and $\mathcal{M} = (M, \langle s^{\mathcal{M}} \mid s \in \mathcal{L}\rangle)$ is an $\mathcal{L}$-structure.

(1) $M \restriction \mathcal{K} = (M, \langle s^{\mathcal{M}} \mid s \in \mathcal{K}\rangle)$ is called a *reduct* of $\mathcal{M}$, more precisely the *reduct* of $\mathcal{M}$ to the language $\mathcal{K}$.
(2) $\mathcal{M}$ is called an *expansion* of $\mathcal{M} \restriction \mathcal{K}$. In other words, $\mathcal{M}$ is an expansion of a structure $\mathcal{N}$ if $\mathcal{N}$ is a reduct of $\mathcal{M}$.

Example 1.1.9.

(1) $\mathcal{M} = (\mathbb{R}, 0^{\mathbb{R}}, 1^{\mathbb{R}}, +^{\mathbb{R}}, \cdot^{\mathbb{R}}, <^{\mathbb{R}})$ is an $\mathcal{L}_{OF}$-structure (in fact it is an ordered field, i.e. it satisfies the axioms of ordered fields), and $\mathcal{M} \restriction \mathcal{L}_R = (\mathbb{R}, 0^{\mathbb{R}}, 1^{\mathbb{R}}, +^{\mathbb{R}}, \cdot^{\mathbb{R}})$ is its reduct to the language $\mathcal{L}_R$ of rings.

(2) Suppose that $\mathcal{M} = (M, \langle s^{\mathcal{M}} \mid s \in \mathcal{L}\rangle)$ is an is an $\mathcal{L}$-structure and $A \subseteq M$. Then $\mathcal{M}_A = (M, \langle s^{\mathcal{M}} \mid s \in \mathcal{L}\rangle \cup \langle a \mid a \in A\rangle)$ is an expansion of $\mathcal{M}$ to the language $\mathcal{L}_A = \mathcal{L} \cup A$.

The structure $\mathcal{M}_A$ has the property that every homomorphism $h \colon \mathcal{M}_A \to \mathcal{M}_A$ fixes A pointwise.

We now begin with building formulas from the language/alphabet. One first builds terms, and from those, formulas. The notion of *term* generalises the notion of polynomials over $(\mathbb{Z}, 0, 1, +, \cdot)$. The terms in the language of rings are the polynomials with coefficients in $\mathbb{Z}$ in an arbitrary number of variables.

We fix a sequence $\langle v_n \mid n \in \mathbb{N}\rangle$ of variables once and for all. We will still use the notation x, y, z for variables, but this will mean that they are among the v_n.

In the following, a word with n letters from a set S is formally a function $f \colon \{0, \dots, n-1\} \to S$. If S is countable, one can assume that S is a set of natural numbers, to realise words a partial functions on $\mathbb{N}$.

Definition 1.1.10. The following words are $\mathcal{L}$-terms:

(1) Every variable v_n
(2) Every constant symbol in $\mathcal{L}$
(3) $f(t_0, \dots, t_{n-1})$, if f is an n-ary function symbol and $t_0, \dots, t_{n-1}$ are $\mathcal{L}$-terms

The $\mathcal{L}$-terms are those words generated by the rules (1)-(3).

We next list the logical symbols, that are allowed independent of the language.

Definition 1.1.11. The following symbols are called *logical symbols*:

(1) Variables $v_n \in \mathrm{Var}$
(2) The equality symbol $\doteq$
(3) The negation symbol $\neg$
(4) The disjunction symbol $\vee$
(5) The existential quantifier $\exists$
(6) The left bracket (right bracket) and comma ,[2]

I replaced the original $\wedge$ by $\vee$.

$\mathcal{L}$-formulas are, informally, those words that make sense. They are built as follows.

Definition 1.1.12. The following words are $\mathcal{L}$-formulas:

(1) $s \doteq t$, if s, t are $\mathcal{L}$-terms.
(2) $R(t_0, \dots, t_{k-1})$, if R is a k-ary relation symbol and $t_0, \dots, t_{k-1}$ are terms
(3) $(\neg\varphi)$, if φ is an $\mathcal{L}$-formula
(4) $(\varphi \vee \psi)$, if φ, ψ are $\mathcal{L}$-formulas
(5) $(\exists x \varphi)$, if φ is an $\mathcal{L}$-formula and x is a variable

$\mathcal{L}$-formulas are those words generated by the rules (1)-(5). Moreover, a formula is called *quantifier-free* if it is generated using only (1)-(4), and *atomic* if it is generated only from (1) and (2).

While this is the formal definition of formulas, we will always allow the usual abbreviations to simplify the notation. For example, we write $x + y$ for $+(x, y)$ or abbreviate $((x < y) \wedge (y < z))$ by $x < y < z$. We also leave out brackets when there is no danger of confusion.

Note that in the previous definition, the brackets around $\varphi \wedge \psi$ are necessary, since one could otherwise not distinguish between $\exists x(\varphi \wedge \psi)$ and $(\exists x\ \varphi) \wedge \psi$. The brackets around $\neg\varphi$ and $\exists x\varphi$ are not strictly necessary:[3] one could still prove Lemma 1.1.17, but not Lemma 1.1.15.

[2] Some authors call these *auxiliary* symbols instead of logical symbols.
[3] For example, there are no brackets there in Martin Ziegler's book.

We will often do induction on terms. This means that we show a statement for variables and constants in the beginning of the induction, and show that it holds for $f(t_0, \dots, t_k)$ assuming it holds for $t_0, \dots, t_k$ in the induction step. This is a valid induction on $n \in \mathbb{N}$, since the term $f(t_0, \dots, t_k)$ has some length n, while the subterms $t_0, \dots, t_k$ are strictly shorter.

Induction on formulas works similarly.

The disjunction $\vee$ and the universal quantifier $\forall$ are still missing. It is convenient to introduce them as notations, rather than as a part of the language itself, since this reduces the number of cases in proofs. We will call this the *extended language*, and will always use it from now on.

Notation 1.1.13. (Extended language)

(1) $(\varphi \wedge \psi) := (\neg(\neg\varphi \vee \neg\psi))$
(2) $(\varphi \to \psi) := ((\neg\varphi) \vee \psi))$
(3) $(\varphi \leftrightarrow \psi) := ((\varphi \to \psi) \wedge (\psi \to \varphi))$
(4) $(\forall x\ \varphi) := (\neg(\exists x\ \neg\varphi))$
(5) $(\varphi_0 \wedge \dots \wedge \varphi_n) := \underbrace{((((}_{n}\varphi_0 \wedge \varphi_1) \wedge \varphi_2) \cdots \wedge \varphi_n)$
(6) $(\varphi_0 \vee \dots \vee \varphi_n) := \underbrace{((((}_{n}\varphi_0 \vee \varphi_1) \vee \varphi_2) \cdots \vee \varphi_n)$
(7) $(\exists x_0 \dots x_n\ \varphi) := \exists x_0(\exists x_1(\dots(\exists x_n \varphi \underbrace{)))}_{n}$
(8) $(\forall x_0 \dots x_n\ \varphi) := \forall x_0(\forall x_1(\dots(\forall x_n \varphi \underbrace{)))}_{n}$

The following is the usual formulation of the group axioms in the extended language, using some abbreviations.

Example 1.1.14. The group axioms are the following formulas in the language $\mathcal{L}_G$:

(1) $\forall x, y, z\ (x \cdot y) \cdot z \doteq x \cdot (y \cdot z)$
(2) $\forall x\ (x \cdot 1 \doteq 1 \cdot x \doteq x)$
(3) $\forall x\ (x \cdot x^{-1} \doteq x^{-1} \cdot x \doteq 1)$

Formally, a word is a sequence of symbols in a set S, or in other words, a function $f\colon \{0, \dots, n\} \to S$. If $f\colon \{0, \dots, n\} \to S$ is a word, then an *initial segment* is a restriction $f{\restriction}\{0, \dots, k\}$ for some $k \leq n$. An *end segment* is defined similarly.

Lemma 1.1.15.

(1) *An $\mathcal{L}$-term cannot be a proper inital segment or end segment of another $\mathcal{L}$-term.*
(2) *An $\mathcal{L}$-formula cannot be a proper initial segment or end segment of another $\mathcal{L}$-formula.*

Proof. (1): It is a bit easier than the following argument to see this by observing that the left and right brackets in a term cancel out, so given the beginning of a term, one can uniquely determine its end. So the next argument is not necessary, but I left it here.

Recall that in $f(t_0, \dots, t_m)$, f, (,), , are symbols and the t_i are themselves words. If $f(t_0, \dots, t_m)$ is an initial segment or end segment of $g(u_0, \dots, u_n)$, then it is easy to see from the inductive hypothesis that $m = n$, $t_i = u_i$ for all $i \leq m$ and $f = g$. We only consider one case in detail, since the other cases use similar steps.

Suppose that $s = f(s_0, \dots, s_k)$ and $u = g(u_0, \dots, u_l)$ are $\mathcal{L}$-terms and s is an initial segment of u. We will see that $s = t$. Since the first symbols of s and t agree, we have $f = g$, and since f is a k-ary function symbol, so is g, and hence $k = l$. We now show by induction that $s_i = t_i$ for all $i \leq k$. Write $u \sqsubseteq v$ if u is an initial segment of v, and $u \sqsubset v$ if it is a proper initial segment. Either $s_0 \sqsubset t_0$, $t_0 \sqsubset s_0$, or $s_0 = t_0$. The first two cases are impossible by the inductive hypothesis, so $s_0 = t_0$. Moving on to the next term, we have either $s_1 \sqsubset t_1$, $t_1 \sqsubset s_1$, or $s_1 = t_1$, etc.

(2): Suppose that ψ is a proper initial segment or end segment of θ. When θ is atomic, i.e. of the form $s \doteq t$ or $R(t_0, \ldots, t_n)$, then the claim follows from (1). When θ equals $(\neg\varphi)$, $(\varphi \wedge \psi)$ or $(\exists x\ \varphi)$, it is easy to see that the claim follows from the inductive hypothesis (2). □

Lecture 2
14. April

Note that we often do an induction on the length of formulas. A more interesting notion of measuring the size of terms and formulas is their *depth*, where, informally, each step in the construction of a term of formula adds 1 to their depth. All proofs would work for induction on the depth as well.

A *segment* of a word $f\colon \{0, \ldots, n\} \to S$ is a connected subword g of f, i.e., there are $k, l \in \mathbb{N}$ such that $g\colon \{0, \ldots, k\} \to S$, $g(i) = f(l + i)$ for all $i \leq k$.

Definition 1.1.16. A *subformula* φ of an $\mathcal{L}$-formula ψ is a segment of ψ that is itself an $\mathcal{L}$-formula. It is a *proper subformula* if additionally $\varphi \neq \psi$.

Lemma 1.1.17. [4] *All subformulas of a formula φ appear in its construction, i.e.*

1. *Atomic formulas $s \doteq t$ and $R(t_0, \ldots, t_k)$ do not have any proper subformulas.*
2. *Any proper subformula of*
   - (a) *$(\neg\varphi)$ is a subformula of φ;*
   - (b) *$(\varphi \vee \psi)$ is a subformula of φ or a subformula of ψ;*
   - (c) *$(\exists x\ \varphi)$ is a subformula of φ.*

Therefore, for each nonatomic formula φ, there is a unique way in which φ is built from one or two other formulas.

Proof. This follows from Lemma 1.1.15. □

The previous lemma shows that one can recover the way in which the formula was built. In particular, this shows that one has avoided ambiguous formulas such as $\exists x\ \varphi \wedge \psi$, which could have meant either $\exists x\ (\varphi \wedge \psi)$ or $(\exists x\ \varphi) \wedge \psi$

The role of variables is relevant for formal derivations later on. It is important to distinguish between *free* and *bound* variables. For example, the variable x is free in $x < y$, but is bound by the quantifier $\forall x$ in $\forall x\ x < y$.

Definition 1.1.18. An occurence of a variable x in an $\mathcal{L}$-formula θ is *free* if this occurence is not bound by a quantifier, i.e.:

(a) If θ is an atomic formula, then every occurence of x is free.
(b) If θ is the formula $(\varphi \wedge \psi)$, then an occurence of x in φ is *free* in θ if it is free in φ; the same holds for ψ.
(c) If θ is the formula $(\exists y\ \varphi)$, then an occurence of x in φ is free in θ if it is free in φ and $x \neq y$.

An occurence of a variable x in an $\mathcal{L}$-formula is *bound* if it is not free.

1.2. **Semantics.** We now define when a formula is true in a structure, i.e. the *semantics*, or meaning, of the formula in the structure. The definition takes as inputs two objects, a structure $\mathcal{M}$ and a formula φ, and outputs whether the formula holds in the structure. One writes $\mathcal{M} \models \varphi$ if φ holds in $\mathcal{M}$, i.e. $\mathcal{M}$ is a model of φ.

Note that there is a difference between formal statements and their truth within a structure (defined formally by semantics), and informal mathematical statements that describe the structure from the outside. For example, the size of an infinite structure is a property that can be seen in the mathematical universe. E.g. the field $\mathbb{C}_{\text{alg}}$ of algebraic

[4] This also holds for the extended language, by the same argument.

complex numbers is countable, but the field $\mathbb{C}$ of complex numbers is uncountable. However, this cannot be expressed within the structures, since $\mathbb{C}_{\text{alg}}$ and $\mathbb{C}$ satisfy precisely the same formulas ($\mathbb{C}_{\text{alg}} \prec \mathbb{C}$ is an elementary substructure, as we will see later).

The formula $\forall x, y, z\ ((x \cdot y) \cdot z = x \cdot (y \cdot z))$ holds in an $\mathcal{L}_G$-structure $(G, 1, \cdot, ^{-1})$, if for all $a, b, c \in G$, the formula $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ holds with the values a, b, c assigned to x, y, z, respectively. We need such assignments for giving a recursive definition of validity of a formula in a structure.

Let $\text{Var} = \{v_n \mid n \in \mathbb{N}\}$ always denote our fixed set of variables.

Definition 1.2.1. An *assignment* (of variables) for a structure $\mathcal{M} = (M, \mathcal{F})$ is a function $\xi\colon \text{Var} \to M$.

Definition 1.2.2. Suppose $\mathcal{M} = (M, \mathcal{F})$ is an $\mathcal{L}$-structure and ξ is an assignment for $\mathcal{M}$. We define $t^{\mathcal{M},\xi}$ by induction on $\mathcal{L}$-terms:

(1) $c^{\mathcal{M},\xi} = c^{\mathcal{M}}$, if $c \in \mathcal{L}$ is a constant symbol
(2) $v_i^{\mathcal{M},\xi} = \xi(v_i)$ for all variables v_i
(3) $f(t_0, \ldots, t_{k-1})^{\mathcal{M},\xi} = f^{\mathcal{M}}(t_0^{\mathcal{M},\xi}, \ldots, t_{k-1}^{\mathcal{M},\xi})$ if f is a k-ary function symbol

Example 1.2.3. For $\mathcal{L} = \{0, 1, +, \cdot\}$, $(\mathbb{Q}, 0^{\mathbb{Q}}, 1^{\mathbb{Q}}, +^{\mathbb{Q}}, \cdot^{\mathbb{Q}})$, the polynomial $t = (v_0 \cdot v_0) + (v_1 \cdot v_2)$ and the assignment $\xi(v_i) = i + 2$, we have $t^{\mathcal{M},\xi} = 16$.

Lemma 1.2.4. *Suppose $\mathcal{M} = (M, \mathcal{F})$ is an $\mathcal{L}$-structure and t is an $\mathcal{L}$-term. Then $t^{\mathcal{M},\xi}$ depends only on the values $\xi(v_i)$ for variables v_i that appear in t.*

Proof. This is immediate, since the value $\xi(v_i)$ appears in the definition of $t^{\mathcal{M},\xi}$ only if v_i appears in t.

More formally, we show by induction on $\mathcal{L}$-terms t that for assignments ξ and ζ for $\mathcal{M}$ such that $\xi(v_i) = \zeta(v_i)$ for all variables v_i that appear in t, we have $t^{\mathcal{M},\xi} = t^{\mathcal{M},\zeta}$:

(1) For variables v_n, $v_n^{\mathcal{M},\xi} = \xi(v_n) = \zeta(v_n) = v_n^{\mathcal{M},\zeta}$.
(2) For constants c, $c^{\mathcal{M},\xi} = c^{\mathcal{M}} = c^{\mathcal{M},\zeta}$.
(3) If $f \in \mathcal{L}$ is a k-ary function symbol and $t_0, \ldots, t_{k-1}$ are terms, then $f(t_0, \ldots, t_{k-1})^{\mathcal{M},\xi} = f(t_0^{\mathcal{M},\beta}, \ldots, t_{k-1}^{\mathcal{M},\xi}) = f(t_0^{\mathcal{M},\zeta}, \ldots, t_{k-1}^{\mathcal{M},\zeta} = f(t_0, \ldots, t_{k-1})^{\mathcal{M},\zeta}$ by the inductive hypothesis.

□

Notation 1.2.5.

(1) If t is an $\mathcal{L}$-term, we write $t = t(x_0, \ldots, x_{n-1})$ if $x_0, \ldots, x_{n-1}$ lists all variables in t in the order of their first appearance in φ.
(2) For an $\mathcal{L}$-term $t = t(x_0, \ldots, x_{n-1})$ and an assignment ξ for $\mathcal{M}$ with $\xi(x_i) = a_i$ for $i < n$, we write $t^{\mathcal{M},a_0,\ldots,a_{n-1}}$ for $t^{\mathcal{M},\xi}$.

To define when a formula is true in a structure, we will need to inductively add more values to an assignment:

Definition 1.2.6. Suppose that ξ is an assignment for $\mathcal{M} = (M, \mathcal{F})$, x is a variable and $a \in M$. The assignment $\xi\frac{a}{x}$ is defined by

$$\xi\frac{a}{x}(y) = \begin{cases} a & \text{if } x = y \\ \xi(y) & \text{if } x \neq y. \end{cases}.$$

Here is the definition of truth in a structure, just as you would expect:

Definition 1.2.7. Suppose that ξ is an assignment for an $\mathcal{L}$-structure $\mathcal{M} = (M, \mathcal{F})$. We define the statement φ *holds in* $\mathcal{M}$ *for* ξ, written as $\mathcal{M} \models \varphi[\xi]$, by induction on $\mathcal{L}$-formulas φ:

(1) $\mathcal{M} \models s \doteq t\ [\xi] \iff s^{\mathcal{M},\xi} = t^{\mathcal{M},\xi}$.

(2) $\mathcal{M} \models R(t_0, \ldots, t_k)[\xi] \iff R^{\mathcal{M}}(t_0^{\mathcal{M},\xi}, \ldots, t_k^{\mathcal{M},\xi})$.
(3) $\mathcal{M} \models (\neg\psi)[\xi] \iff \mathcal{M} \not\models \psi[\xi]$.
(4) $\mathcal{M} \models (\psi \wedge \theta)[\xi] \iff \mathcal{M} \models \psi[\xi]$ and $\mathcal{M} \models \theta[\xi]$.
(5) $\mathcal{M} \models (\exists x\ \psi)[\xi] \iff \exists a \in M\ \mathcal{M} \models \psi[\xi\frac{a}{x}]$.

1.3. **Elementary substructures.** The notion of substructure was introduced in Definition 1.1.7 above. The next lemma shows that every structure has a smallest substructure.

Lemma 1.3.1. *Suppose that $\mathcal{N} = (N, \langle s^{\mathcal{N}} \mid s \in \mathcal{L}\rangle)$ is an $\mathcal{L}$-structure and $A \subseteq M$. The following conditions are equivalent:*

(a) *There is a substructure $\mathcal{A}$ of $\mathcal{N}$ of the form $\mathcal{A} = (A, \langle s^{\mathcal{A}} \mid s \in \mathcal{L}\rangle)$.*
(b) *For any $a_0, \ldots, a_n \in A$ and all $\mathcal{L}$-terms $t = t(x_0, \ldots, x_n)$, we have $t^{\mathcal{N}, a_0, \ldots, a_n} \in A$.*

Assuming that $\mathcal{L}$ contains at least one constant symbol,[5] *it follows that there is a ($\subseteq$-)least substructure of $\mathcal{N}$, and its domain is $\{t^{\mathcal{N}, a_0, \ldots, a_n} \mid t(x_0, \ldots, x_n)$ is an $\mathcal{L}$-Term and $a_0, \ldots, a_n \in A\}$.*

Proof. Exercise ☐

A substructure can have very different properties than the original structure. For instance $(\mathbb{Z}, 0^{\mathbb{Z}}, 1^{\mathbb{Z}}, +^{\mathbb{Z}}, \cdot^{\mathbb{Z}})$ is a substructure of $(\mathbb{R}, 0^{\mathbb{R}}, 1^{\mathbb{R}}, +^{\mathbb{R}}, \cdot^{\mathbb{R}})$, so a substructure of a field is not necessarily a field. The next definition describes a more useful notion.

We here already use the notation $\mathcal{M} \models \varphi[a]$ that is only introduced after Lemma 1.3.4 below.

Definition 1.3.2. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures.

(1) $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$, written as $\mathcal{M} \prec \mathcal{N}$, if $M \subseteq N$ and for all $a_0, \ldots, a_{n-1} \in M$ and all $\mathcal{L}$-formulas φ with n free variables
$$\mathcal{M} \models \varphi[a_0, \ldots, a_{n-1}] \iff \mathcal{N} \models \varphi[a_0, \ldots, a_{n-1}].$$
(2) An $\mathcal{L}$-*sentence* is an $\mathcal{L}$-formula without free variables.
(3) $\mathcal{M}$ and $\mathcal{N}$ are called *elementary equivalent* if for all $\mathcal{L}$-sentences φ,
$$\mathcal{M} \models \varphi \iff \mathcal{N} \models \varphi.$$

Example 1.3.3.

(1) Every substructure of a complete graph (i.e., there is an edge between any two vertices) is itself a complete graph. If both are infinite, it is also an elementary substructure. (We will prove this later in the lecture.)
(2) $(\mathbb{Z}, \leq^{\mathbb{Z}})$ is a substructure of $(\mathbb{Q}, \leq^{\mathbb{Q}})$, but not an elementary substructure. (Consider the formula $\exists x\ \exists y\ (\forall z(x \not\leq z \wedge z \not\leq y) \wedge x \leq y \wedge x \neq y)$.)
(3) $(2\mathbb{N}, +^{\mathbb{N}}, 0^{\mathbb{N}})$ is not an elementary substructure of $(\mathbb{N}, +^{\mathbb{N}}, 0^{\mathbb{N}})$. (We can look at the notion of evenness ($\exists y\ x = y + y$). In $\mathbb{N}$, $1 + 1$ is even, but this fails in $2\mathbb{N}$.)
(4) $(\mathbb{Q}, \leq^{\mathbb{Q}})$ is an elementary substructure of $(\mathbb{R}, \leq^{\mathbb{R}})$. (We will prove this later in the lecture.)

Lecture 3
19. April

The next lemma was already used in Definition 1.3.2.

Lemma 1.3.4. *If ξ, ζ are assignments for an $\mathcal{L}$-structure $\mathcal{M}$ such that $\xi(v_i) = \zeta(v_i)$ for all variables v_i that are free in φ, then*
$$\mathcal{M} \models \varphi[\xi] \iff \mathcal{M} \models \varphi[\zeta].$$

If $x_0, \ldots, x_n$ are the free variables of φ and $\xi(x_i) = a_i$ for all $i \leq n$, we can therefore write $\mathcal{M} \models \varphi[a_0, \ldots, a_n]$ instead of $\mathcal{M} \models \varphi[\xi]$. If φ has no free variables, we simply write $\mathcal{M} \models \varphi$.

[5] Recall that structures are by definition nonempty. If $\mathcal{L}$ does not contain constant symbols, then the following set is empty.

Proof. This is immediate because only the values of variables occuring in φ appear in the inductive definition of $\mathcal{M} \models \varphi[\xi]$.

In more detail, we do an induction on formulas φ. The induction hypothesis states that the claim holds for φ for all assignments. If $R \in \mathcal{L}$ is an n-ary relation symbol, $t_0, \dots, t_n$ are $\mathcal{L}$-terms and $\varphi = R(t_0, \dots, t_n)$, then

$$\mathcal{M} \models \varphi[\xi] \iff R^{\mathcal{M}}(t_0^{\mathcal{M},\xi}, \dots, t_n^{\mathcal{M},\xi}) \iff R^{\mathcal{M}}(t_0^{\mathcal{M},\zeta}, \dots, t_n^{\mathcal{M},\zeta}) \iff \mathcal{M} \models \varphi[\zeta].$$

The cases $s \doteq t$, $(\neg\psi)$ and $(\psi \wedge \theta)$ are similar. If $\varphi = (\exists x\ \psi)$, then

$$\mathcal{M} \models \varphi[\xi] \iff \exists a \in M\ \mathcal{M} \models \psi[\xi\frac{a}{x}] \iff \exists a \in M\ \mathcal{M} \models \psi[\zeta\frac{a}{x}] \iff \mathcal{M} \models \varphi[\zeta].$$

□

Example 1.3.5. For each $n \in \mathbb{N}$, there is a sentence φ such that $\mathcal{M} = (M, \mathcal{F}) \models \varphi$ if and only if $|M| = n$. E.g. for $n = 3$, let φ be the sentence $\exists x_0, x_1, x_2\ (x_0 \neq x_1 \wedge x_0 \neq x_2 \wedge x_1 \neq x_2 \wedge \forall y\ (y \doteq x_0 \vee y \doteq x_1 \vee y \doteq x_2))$.

Hence a finite structure does not have proper elementary substructures.

Example 1.3.6. If $\mathcal{M} = (M, <^{\mathcal{M}})$ is an elementary substructure of $\mathcal{N} = (\mathbb{N}, <^{\mathbb{N}})$, then $\mathcal{M} = \mathcal{N}$.

Proof. We show $n \in M$ for all $n \in \mathbb{N}$ by induction on n.

To see that $0 \in M$, note that the statement $\exists y \neg\exists x\ (x < y)$ holds in $\mathcal{N}$ and thus it also holds in $\mathcal{M}$. So there is some $a \in M$ with $\mathcal{M} \models \neg\exists x\ (x < y)[a]$. Since $\mathcal{M} \prec \mathcal{N}$, we also have $\mathcal{N} \models \neg\exists x\ (x < y)[a]$. Thus a is the $<^{\mathbb{N}}$-least element of $\mathbb{N}$, i.e. $a = 0$. So $0 = a \in M$.

Now assume that $n \in M$. We have $\mathcal{N} \models \exists z > x\ \neg\exists y(x < y < z))[n]$ (this formula is an abbreviation for $\exists z(z > x \wedge \neg\exists y((x < y) \wedge (z < x)))$). A similar argument as for 0 shows that $n + 1 \in N$. □

Analogous to the notation for terms in Notation 1.2.5, we write $\varphi(x_0, \dots, x_n)$ if $x_0, \dots, x_n$ are precisely the free variables of φ, listed by the first appearance in φ.

Lemma 1.3.7. *(Tarski's test) Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures. The following conditions are equivalent:*

(1) *$\mathcal{M}$ is an elementary substructure of $\mathcal{M}$, i.e. $\mathcal{M} \prec \mathcal{N}$.*
(2) *$\mathcal{M}$ is a substructure of $\mathcal{N}$, and for all $\mathcal{L}$-formulas $\varphi(x, x_0, \dots, x_n)$ and all $a_0, \dots, a_n \in M$:*

If there is some $b \in N$ with $\mathcal{N} \models \varphi[b, a_0, \dots, a_n]$,
then there is some $a \in M$ with $\mathcal{N} \models \varphi[a, a_0, \dots, a_n]$.

Proof. (1)⇒(2): If $\mathcal{N} \models \varphi[b, a_0, \dots, a_n]$, then $\mathcal{N} \models \exists x\ \varphi(x, x_0, \dots, x_n)[a_0, \dots, a_n]$. Since $M \prec N$, there is some $a \in M$ with $\mathcal{M} \models \varphi[a, a_0, \dots, a_n]$. Since $M \prec N$, we have $\mathcal{N} \models \varphi[a, a_0, \dots, a_n]$.

(2)⇒(1): By induction on formulas φ. The cases $\vee$ and $\neg$ are easy.

For the existential case, first suppose that $\varphi = \varphi(x, x_0, \dots, x_n)$ and $\mathcal{M} \models \exists x\ \varphi[a_0, \dots, a_n]$. Then there is some $a \in M$ with $\mathcal{M} \models \varphi[a, a_0, \dots, a_n]$. By the inductive hypothesis for φ, $\mathcal{N} \models \varphi[a, a_0, \dots, a_n]$.

Now suppose that $\mathcal{N} \models \exists x\ \varphi\ [a_0, \dots, a_n]$. By (2), there is some $a \in M$ with $\mathcal{N} \models \varphi[a, a_0, \dots, a_n]$. By the inductive hypothesis for φ, we have $\mathcal{M} \models \varphi[a, a_0, \dots, a_n]$. So $\mathcal{M} \models \exists x\ \varphi\ [a_0, \dots, a_n]$.

□

Homomorphisms (see Definition 1.1.6) preserve interpretations of terms:

Lemma 1.3.8. *Suppose that $\mathcal{M} = (M, \mathcal{F})$, $\mathcal{N} = (N, \mathcal{G})$ are $\mathcal{L}$-structures and $h\colon \mathcal{M} \to \mathcal{N}$ is a homomorphism. Then for any term $t = t(x_0, \dots, x_n)$ and all $a_0, \dots, a_n \in M$,*

$$h(t^{\mathcal{M}, a_0, \dots, a_n}) = t^{\mathcal{N}, h(a_0), \dots, h(a_n)}.$$

Proof. Exercise in the tutorials. □

Isomorphisms preserve the truth of formulas:

Lemma 1.3.9. *Suppose that* $\mathcal{M} = (M, \mathcal{F})$, $\mathcal{N} = (N, \mathcal{G})$ *are* $\mathcal{L}$*-structures and* $h \colon \mathcal{M} \to \mathcal{N}$ *is an isomorphism. Then for any formula* $\varphi = \varphi(x_0, \ldots, x_n)$ *and all* $a_0, \ldots, a_n \in M$*,*

$$\mathcal{M} \models \varphi[a_0, \ldots, a_n] \Longleftrightarrow \mathcal{N} \models \varphi[h(a_0), \ldots, h(a_n)].$$

Proof. Homework exercise. □

1.4. **Theories and axioms.** One often wants to derive results about a structure from *axioms.* A set of $\mathcal{L}$-sentences (formulas with no free variables) is called a *theory.* The sentences in a theory T are often called axioms and T is called an axiom system.

Definition 1.4.1. One says that an $\mathcal{L}$-structure $\mathcal{M}$ *satisfies* an $\mathcal{L}$-theory T, or $\mathcal{M}$ is a *model* of T, in symbols $\mathcal{M} \models T$, if $\mathcal{M} \models \varphi$ for all $\varphi \in T$.

Given an axiom system T, we can ask:

(1) (Syntactic) Which formulas are provable from T? This will be made precise using a proof calculus in Section 1.5.
(2) (Semantic) Which formulas does T imply? (See Definition 1.4.7.) This is equivalent to the previous question by Gödel's proof of the completeness of the proof calculus in chapter 3.

    Which models does T have? We study this question throughout the lecture. We have already looked at the notion of elementary substructure, where one has two structures with the same theory. The question how many models (of a given size) T has is also connected with incompleteness of T, since an axiom system that implies neither φ nor $\neg\varphi$ has some models that satisfy φ and some that don't.

We now look at some examples of theories. *Peano arithmetic* is an axiom system in the language $\mathcal{L}_{\text{Arith}} = \{0, S, +, \cdot\}$ of arithmetic that holds in the structure $(\mathbb{N}, 0^{\mathbb{N}}, S^{\mathbb{N}}, +^{\mathbb{N}}, \cdot^{\mathbb{N}})$ of the natural numbers, where $S^{\mathbb{N}}$ denotes the successor function.

Example 1.4.2. (Peano Arithmetic) PA consists of the axioms:

(1) $\forall x \ (S(x) \neq 0)$
(2) $\forall x, y \ (S(x) \doteq S(y) \to x \doteq y)$
(3) $\forall x, y \ (x + 0 \doteq x)$
(4) $\forall x, y \ (S(x + y) \doteq x + S(y))$
(5) $\forall x, y \ (x \cdot 0 \doteq 0)$
(6) $\forall x, y \ (x \cdot S(y) \doteq x \cdot y + x)$
(7) (Axiom scheme of induction) If $\varphi(x, \vec{y})$ is an $\mathcal{L}_{\text{Arith}}$-formula, then

$$(\varphi(0, \vec{y}) \wedge \forall x \ [\varphi(x, \vec{y}) \to \varphi(S(x), \vec{y})]) \to \forall x \ \varphi(x, \vec{y}).$$

(The axioms (1)-(6) together with the axiom $\forall y \ (y = 0 \vee \exists x \ (S(x) = y))$, which follows from PA , are called Robinson Arithmetic.)

Note that PA consists of infinitely many axioms. (By a theorem of Ryll-Nardzewski, one cannot axiomatise PA with only finitely many axioms.)

Every model of PA satisfies statements about $+$ and $\cdot$ that can be proved by induction, for example division with remainder.

Example 1.4.3. The following statements hold in every model of PA:

(1) $\forall y \ (y = 0 \vee \exists x \ (S(x) = y))$
(2) $\forall x, y, z \ ((x + y) + z = x + (y + z))$
(3) $\forall x, y \ (x + y = y + x)$

In the case of PA, the axioms aim to describe a single structure. Other axioms, e.g. the group axioms, aim to describe a class of structures.

By a class, we mean the collection of all objects with a certain property, for example the class of all sets or the class of all groups. This will be studied in more detail in the next chapter.

Definition 1.4.4. For a class $\mathcal{C}$ of $\mathcal{L}$-structures, an $\mathcal{L}$-theory T is called an *axiomatisation* of $\mathcal{C}$ if $\mathcal{C}$ is the class of $\mathcal{L}$-structures $\mathcal{M}$ with $\mathcal{M} \models T$. $\mathcal{C}$ is called *axiomatisable (resp., finitely axiomatisable)* if it has an axiomatisation (resp., a finite axiomatisation).

Here are examples of axiomatisations of various classes of structures.

Example 1.4.5.

(1) For any language $\mathcal{L}$ and any $n \in \mathbb{N}$ with $n \geq 1$, the class $\mathcal{C}_{\leq n}$ of $\mathcal{L}$-structures with at most elements is axiomatised by the axiom

$$\varphi_{\leq n} = \exists x_0 \ldots \exists x_{n-1}\ \forall y\ \bigvee_{i=0}^{n-1} y \doteq x_i.^{6}$$

Similarly, the class $\mathcal{C}_{\geq n}$ of $\mathcal{L}$-structures with at least elements can be axiomatised by the axiom

$$\varphi_{\geq n} = \exists x_0 \ldots \exists x_{n-1} \bigwedge_{i<j\leq n-1} \neg(x_i \doteq x_j).$$

(2) For any language $\mathcal{L}$, the class $\mathcal{C}_\infty$ of infinite $\mathcal{L}$-stuctures is axiomatised by the theory

$$T_\infty = \{\varphi_{\geq n} \mid n \in \mathbb{N}\}.$$

We will see later that $\mathcal{C}_\infty$ has no finite axiomatisation.

Let $\mathcal{C}_{\text{fin}}$ denote the class of finite $\mathcal{L}$-structures. We will see later that $\mathcal{C}_{\text{fin}}$ cannot be axiomatised at all.

(3) The class of (symmetric) graphs $\mathcal{G} = (G, E^{\mathcal{G}})$ with no cycles is axiomatised in the language $\mathcal{L} = \{E\}$ with a single binary relation symbol by

$$T = \{\varphi\} \cup \{\varphi_n \mid n \in \mathbb{N}\},$$

where $\varphi = (\forall x, y\ (E(x,y) \rightarrow E(y,x)))$ and $\varphi_n = (\forall x_0, \ldots, x_n(x_0 = x_n \rightarrow \neg \bigwedge_{i<n} E(x_i, x_{i+1}))$.

Lecture 4
21. April

Definition 1.4.6. The *theory* $\mathrm{Th}(\mathcal{M})$ of an $\mathcal{L}$-structure $\mathcal{M}$ is defined as the set of $\mathcal{L}$-sentences φ with $\mathcal{M} \models \varphi$.

We already introduced $\models$ for truth of a formula in a model (with an assignment). One also writes $\models$ for (semantical) implication:

Definition 1.4.7. Suppose that T is an $\mathcal{L}$-theory and φ is an $\mathcal{L}$-formula.

T (semantically) *implies* φ, written as $T \models_{\mathcal{L}} \varphi$, if every model of T, with any assignment, is a model of φ.

Moreover $\models_{\mathcal{L}} \varphi$ means that φ is *universally valid*, or *universally true*, i.e. φ holds in any $\mathcal{L}$-structure with any assignment.

A first observation is that implication does not depend on the language.

---

[6]This is an abbreviation for $y = x_0 \vee \cdots \vee y = x_n$.

Lemma 1.4.8. *Suppose that $\mathcal{K} \subseteq \mathcal{L}$ are languages, T is a $\mathcal{K}$-theory and φ is a $\mathcal{K}$-formula. Then*

$$T \models_{\mathcal{K}} \varphi \Longleftrightarrow T \models_{\mathcal{L}} \varphi.$$

Proof. For any $\mathcal{K}$-structure $\mathcal{M}$ and any assignment ξ for $\mathcal{M}$, we have $\mathcal{M} \models \varphi[\xi] \Longleftrightarrow \mathcal{M}{\upharpoonright}\mathcal{K} \models \varphi[\xi]$. This is because only symbols in $\mathcal{K}$ are actually used. It is an easy induction on formulas, similar to Lemma 1.3.4.

We can assume that φ is an $\mathcal{K}$-sentence by replacing a formula φ with free variables $x_0, \ldots, x_n$ by $\forall x_0, \ldots, x_n\ \varphi$.

First suppose that $T \models_{\mathcal{K}} \varphi$. If $\mathcal{M}$ is an $\mathcal{L}$-structure with $\mathcal{M} \models T$, then $\mathcal{M}{\upharpoonright}\mathcal{K} \models T$ by the remark in the beginning of the proof. Hence $\mathcal{M}{\upharpoonright}\mathcal{K} \models \varphi$ and $\mathcal{M} \models \varphi$

Now suppose that $T \models_{\mathcal{L}} \varphi$. If $\mathcal{M}$ is an $\mathcal{K}$-structure with $\mathcal{M} \models T$, take any $\mathcal{L}$-structure $\mathcal{N}$ expanding $\mathcal{M}$, i.e. with arbitrary interpretations of the new symbols. (Here we use that $M \neq \emptyset$, since all structures are nonempty by definition.) Then $\mathcal{N} \models T$ by the above remark. Hence $\mathcal{N} \models \varphi$ and $\mathcal{M} = \mathcal{N}{\upharpoonright}\mathcal{K} \models \varphi$. □

Note that a formula $\varphi(x_0, \ldots, x_n)$ is universally valid if and only if its universal closure $\forall x_0, \ldots, x_n\ \varphi(x_0, \ldots, x_n)$ is universally valid.

To study e.g. the class of all groups, one want to determine which $\mathcal{L}_G$-sentences are implied by the group axioms. It is useful to study universal truths, since the implication from φ to ψ is equivalent to universal truth of $\varphi \to \psi$.

1.5. Universal truths and the Hilbert calculus.

A universal truth is an $\mathcal{L}$-formula that is true in any $\mathcal{L}$-structure for any assignment of variables. We will collect several kinds of universal truths and will then build up a proof calculus from them.

It is easy to check that $\varphi \to \varphi$ and $(\varphi \wedge \psi) \vee (\neg\varphi) \vee (\neg\psi)$ universally valid. More generally, for any *Boolean combination* of formulas $\varphi_0, \ldots, \varphi_n$ using $\vee, \neg, \wedge$ and $\to$, the truth of a combination in a model $\mathcal{M}$ depends only on the truth values of $\varphi_0, \ldots, \varphi_n$ in $\mathcal{M}$. (This is easy to see from the definition of $\mathcal{M} \models \varphi$.)

Propositional logic studies this. To define propositional formulas, we fix a countably infinite set $\mathbb{P}$, i.e. with one element for each natural number, whose elements we call *propositional variables*. For example, $p \to p$ for $p \in \mathbb{P}$ is a propositional formula and $\varphi \to \varphi$ is a $\mathcal{L}$-formula obtained by replacing p by φ in $p \to p$. A propositional formula can be understood as a string of symbols, but this does not fit into the framework of languages studied above, and propositional variables are not logical variables as studied above.

Definition 1.5.1.

(1) *Propositional formulas* are formal *Boolean combinations* of propositional variables $p \in \mathbb{P}$, i.e. they are generated as follows:
   (a) Each $p \in \mathbb{P}$ is a propositional formula.
   (b) If p and q are propositional formulas, then $(p \vee q)$ is a propositional formula.
   (c) If p is a propositional formula, then $(\neg p)$ is a propositional formula.

(2) A *propositional assignment* is an arbitrary function $\mu \colon \mathbb{P} \to \{0, 1\}$, where 1 stands for *true* and 0 for *false*. μ can be extended to a function on all propositional formulas by letting
   (a) $\mu(\neg q) = 1$ if $\mu(q) = 0$, and $\mu(\neg q) = 0$ otherwise;
   (b) $\mu(q \vee r) = 1$ if $(\mu(q) = 1$ or $\mu(r) = 1)$, and $\mu(q \vee r) = 0$ otherwise.

(3) A propositional formula p is called a *propositional tautology* if $\mu(p) = 1$ holds for all propositional assignments μ for p.

changed terminology (previously *universally valid*)

We also use the abbreviations $(p \wedge q) = \neg((\neg p) \vee (\neg q))$ and $(p \to q) = ((\neg p) \vee q)$. Using (2), one obtains

(a) $\mu(p \wedge q) = 1$ iff $(\mu(p) = 1$ and $\mu(q) = 1)$.

(b) $\mu(p \to q) = 1$ iff ($\mu(p) = 0$ or $\mu(q) = 1$).

Example 1.5.2. For all propositional variables p and q, the propositional formulas $(p \to p)$ and $((p \wedge (p \to q)) \to q)$ are propositional tautologies.

Definition 1.5.3. A *tautology* is an $\mathcal{L}$-formula that is obtained from a propositional tautology p by replacing each propositional variable p_n in p by an $\mathcal{L}$-formula φ_n.

Lemma 1.5.4. *(Tautologies) All tautologies are universally valid.*

Proof. Suppose that φ is a tautology that arises from a Boolean combination p of propositional variables $p_0, \dots, p_n$ by replacing p_i by the $\mathcal{L}$-formula φ_i for all $i \leq n$. Suppose further that $\mathcal{M}$ is an $\mathcal{L}$-structure and ξ is an assignment for $\mathcal{M}$.

We consider the truth values of subformulas of φ in $\mathcal{M}$ for ξ. If we choose the components p_i as true or false according to these truth values, the truth of subformulas of φ in $\mathcal{M}$ for ξ will correspond to the values of the corresponding propositional subformulas of p. This is because the inductive definition of μ corresponds to the inductive definition of $\mathcal{M} \models \varphi$.

In more detail, we define $\mu(p_i) = 1 \iff \mathcal{M} \models \varphi_i[\xi]$ for $i \leq n$ and let $\mu(q)$ be arbitrary for all other propositional variables q. Using Definitions 1.2.7 and 1.5.1, we see by induction on Boolean combinations that $\mathcal{M} \models \varphi[\xi] \iff \mu(p) = 1$. Since p is a propositional tautology, $\mathcal{M} \models \varphi[\xi]$. □

We will allow tautologies as basic steps in the proof calculus. Note that some authors fix a finite list of tautologies and derive all other ones from them using a proof calculus, see for instance [1, Page 11]. But then even proof of simple statements such as $\varphi \to \varphi$ can be quite complicated, see [1, Page 14].

We next consider universal truths about equality. The next lemma is immediate.

Lemma 1.5.5. *(Axioms of equality) The following $\mathcal{L}$-sentences are universally valid.*

(1) *(Reflexivity)* $\forall x \; x \doteq x$
(2) *(Symmetry)* $\forall x, y \; (x \doteq y \to y \doteq x)$
(3) *(Transitivity)* $\forall x, y \; (x \doteq y \wedge y \doteq z \to x \doteq z)$
(4) *(Congruence for functions) For all n-ary relation symbols f,*

$$\forall x_0, \dots x_n, y_0, \dots, y_n \; ((x_0 \doteq y_0 \wedge \dots \wedge x_n \doteq y_n) \to f(x_0, \dots, x_n) \doteq f(y_0, \dots, y_n)).$$

(5) *(Congruence for relations) For all n-ary relation symbols R,*

$$\forall x_0, \dots x_n, y_0, \dots, y_n \; ((x_0 \doteq y_0 \wedge \dots \wedge x_n \doteq y_n) \to (R(x_0, \dots, x_n) \leftrightarrow R(y_0, \dots, y_n))).$$

The next three lemmas collect ways to generate universal truths.

Lemma 1.5.6. *(Modus ponens) If φ and $\varphi \to \psi$ are universally valid formulas, then ψ is a universally valid formula.*

Proof. Suppose that ξ is an assignment for $\mathcal{M}$. Then both φ and $\varphi \to \psi$ hold in $\mathcal{M}$ for ξ. Recall that $\varphi \to \psi$ is defined as $(\neg\varphi) \vee \psi$, so we have $\mathcal{M} \not\models \varphi[\xi]$ or $\mathcal{M} \models \psi[\xi]$ □

The modus tollens states that if $\neg\psi$ and $\varphi \to \psi$ are universally valid, then $\neg\varphi$ is universally valid. This can be found by using the tautology $(\varphi \to \psi) \longrightarrow (\neg\psi \to \neg\varphi)$ and applying the modus ponens twice.

Lemma 1.5.7. *($\exists^{\to}$ introduction) If $\varphi \to \psi$ is a universally valid formula and x is not free in ψ, then $(\exists x \varphi) \to \psi$ is an universally valid formula.*

changed notation (previously $\exists$-elimination)

Proof. If $\mathcal{M} \models (\exists x \varphi)[\xi]$, then there is some $a \in M$ with $\mathcal{M} \models \varphi[\xi\frac{a}{x}]$. Since $\varphi \to \psi$ is universally valid, we have $\mathcal{M} \models \psi[\xi\frac{a}{x}]$, so $\mathcal{M} \models \psi[\xi]$ by Lemma 1.3.4. □

The next, and final, $^{\rightarrow}\exists$-*axiom* states that certain implications of the form

$$\varphi\frac{t}{x} \rightarrow \exists x\ \varphi$$

are universally valid. Here $\varphi\frac{s}{x}$ means that all free occurences of the variable x in φ are replaced by the term s. Although this definition is clear, we give the full recursive definition in the following, since this definition is used in the next lemmas.

Definition 1.5.8. Suppose that s, t are $\mathcal{L}$-terms and x is a variable. The term $t\frac{s}{x}$ is defined by replacing all occurences of x by s. More formally, define by induction on t:

(1) For $y \in \mathrm{Var}$, $y\frac{s}{x} = \begin{cases} s \text{ if } x = y \\ y \text{ otherwise} \end{cases}$
(2) For constants $c \in \mathcal{L}$, $c\frac{s}{x} = c$.
(3) If $f \in \mathcal{L}$ is an n-ary function symbol and $t_0, \ldots, t_{n-1}$ are $\mathcal{L}$-terms, then $f(t_0, \ldots, t_{n-1})\frac{s}{x} = f(t_0\frac{s}{x}, \ldots, t_{n-1}\frac{s}{x})$.

Suppose that φ is an $\mathcal{L}$-formula, x is a variable and s is an $\mathcal{L}$-term. The formula $\varphi\frac{s}{x}$ is defined by replacing all free occurences of x by s. More formally, define by induction on φ:

(1) $(u \doteq v)\frac{s}{x} = (u\frac{s}{x} \doteq v\frac{s}{x})$ and $R(t_0, \ldots, t_n)\frac{s}{x} = R(t_0\frac{s}{x}, \ldots, t_n\frac{s}{x})$ for terms $u, v, t_0, \ldots, t_n$.
(2) $(\neg\psi)\frac{s}{x} = (\neg(\psi\frac{s}{x}))$.
(3) $(\psi \wedge \theta)\frac{s}{x} = (\psi\frac{s}{x}) \wedge (\theta\frac{s}{x})$.
(4) $(\exists y\ \psi)\frac{s}{x} = \begin{cases} \exists y\ (\psi\frac{s}{x}) \text{ if } x \neq y \\ \exists y\ \psi \text{ if } x = y. \end{cases}$.

The next lemma shows that the interpretation of $t\frac{s}{x}$ can be found by interpreting t, but changing the value at x.

Lecture 5
26. April

Lemma 1.5.9. *(Substitution for terms) For any $\mathcal{L}$-term t and any assignment ξ for an $\mathcal{L}$-structure $\mathcal{M}$,*

$$(t\frac{s}{x})^{\mathcal{M},\xi} = t^{\mathcal{M},(\xi\frac{s^{\mathcal{M},\xi}}{x})}.$$

Proof. By induction on terms.

We have $(x\frac{s}{x})^{\mathcal{M},\xi} = s^{\mathcal{M},\xi} = x^{\mathcal{M},\xi\frac{s^{\mathcal{M},\xi}}{x}}$, if $y \neq x$ is a variable then $(y\frac{s}{x})^{\mathcal{M},\xi} = \xi(y) = y^{\mathcal{M},(\xi\frac{s^{\mathcal{M},\xi}}{x})}$ and if $c \in \mathcal{L}$ is a constant then $(c\frac{s}{x})^{\mathcal{M},\xi} = c^{\mathcal{M}} = c^{\mathcal{M},\xi\frac{s^{\mathcal{M},\xi}}{x}}$.

Moreover, $f(t_0\frac{s}{x}, \ldots, t_n\frac{s}{x})^{\mathcal{M},\xi} = f^{\mathcal{M}}((t_0\frac{s}{x})^{\mathcal{M},\xi}, \ldots, (t_n\frac{s}{x})^{\mathcal{M},\xi}) = f^{\mathcal{M}}(t_0^{\mathcal{M},\xi\frac{s^{\mathcal{M},\xi}}{x}}, \ldots, t_n^{\mathcal{M},\xi\frac{s^{\mathcal{M},\xi}}{x}})$
$= f(t_0, \ldots, t_n)^{\mathcal{M},\xi\frac{s^{\mathcal{M},\xi}}{x}}$ □

When we substitute a variable in a formula, in some cases the formula does not have the intended meaning. This problem is prevented by the next condition.

Definition 1.5.10. A substitution $\varphi\frac{t}{x}$ is *allowed* if no variables of t are bound where x is replaced by t, i.e. at free occurences of x.

Here is a more detailed, recursive definition: the substitution is allowed if

(1) φ is atomic,
(2) $\varphi = (\neg\psi)$ and the substitution $\psi\frac{t}{x}$ is allowed,
(3) $\varphi = (\psi \vee \theta)$ and the substitutions $\psi\frac{t}{x}$ and $\theta\frac{t}{x}$ are allowed,
(4) $\varphi = (\exists y\ \psi)$, the substitution $\psi\frac{t}{x}$ is allowed, and y does not occur in t.

This always holds when t is a constant c. If t is a variable y, then the condition states that y is not bound in the relevant places. The next lemma shows that in this case, substitution works well.

Lemma 1.5.11. *(Substitution for formulas) Suppose that φ is an $\mathcal{L}$-formula and s is an $\mathcal{L}$-term. If the substitution $\varphi\frac{s}{x}$ is allowed, then*

$$\mathcal{M} \models (\varphi\frac{s}{x})[\xi] \iff \mathcal{M} \models \varphi[\xi\frac{s^{\mathcal{M},\xi}}{x}].$$

Proof. If x does not occur freely in φ, then the claim holds by Lemma 1.3.4.

Suppose that x occurs freely in φ. The atomic case follows from Lemma 1.5.9, and the cases $\varphi = (\neg\psi)$ and $\varphi = (\psi \vee \theta)$ are easy.

Suppose that $\varphi = (\exists y\ \psi)$. Since x appears freely in φ, this implies $x \neq y$. Since the substitution is allowed, y does not appear in s. Then:

$$\begin{aligned}\mathcal{M} \models (\exists y\ \psi)\tfrac{s}{x}[\xi] &\iff \mathcal{M} \models \psi\tfrac{s}{x}[\xi\tfrac{a}{y}] \text{ for some } a \in M \\ &\iff \mathcal{M} \models \psi[(\xi\tfrac{a}{y})\tfrac{s^{\mathcal{M},\xi\frac{a}{y}}}{x}] \text{ for some } a \in M \text{ [by the inductive hypothesis]} \\ &\iff \mathcal{M} \models \psi[(\xi\tfrac{a}{y})\tfrac{s^{\mathcal{M},\xi}}{x}] \text{ for some } a \in M \text{ [since } y \text{ does not appear in } s] \\ &\iff \mathcal{M} \models \psi[(\xi\tfrac{s^{\mathcal{M},\xi}}{x})\tfrac{a}{y}] \text{ for some } a \in M \text{ [since } x \neq y] \\ &\iff \mathcal{M} \models (\exists y\ \psi)[\xi\tfrac{s^{\mathcal{M},\xi}}{x}]\end{aligned}$$

Note that the functions $(\xi\frac{a}{y})\frac{b}{x}\colon \mathrm{Var} \to M$ and $(\xi\frac{b}{x})\frac{a}{y}\colon \mathrm{Var} \to M$, where $b = s^{\mathcal{M},\xi}$, are identical since $x \neq y$. □

Lemma 1.5.12. *($^{\to}\exists$-axiom) Suppose that φ is an $\mathcal{L}$-formula, t is an $\mathcal{L}$-term and x is a variable. If the substitution $\varphi\frac{t}{x}$ is allowed, then the formula*

$$\varphi\frac{t}{x} \to \exists x\ \varphi$$

is universally valid.

Proof. Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and ξ is an assignment for $\mathcal{M}$. By Lemma 1.5.11,

$$\mathcal{M} \models \varphi\frac{t}{x}[\xi] \iff \mathcal{M} \models \varphi[\xi\frac{t^{\mathcal{M},\xi}}{x}] \implies \mathcal{M} \models \exists x\ \varphi[\xi].$$

□

We now define the *Hilbert calculus* as the system of formal rules that consists of the above rules to generate universal truths.[7]

Definition 1.5.13. An $\mathcal{L}$-formula φ is called *$\mathcal{L}$-provable* (in the Hilbert calculus) in each of the following cases:

(1) φ is an equality axiom
(2) φ is a tautology
(3) φ is an $^{\to}\exists$-axiom
(4) φ is generated from two $\mathcal{L}$-provable $\mathcal{L}$-formulas using the modus ponens
(5) φ is generated from an $\mathcal{L}$-provable $\mathcal{L}$-formula using the $\exists^{\to}$-rule.

A *formal $\mathcal{L}$-proof* of φ is a list of $\mathcal{L}$-formulas, each of which is $\mathcal{L}$-provable from the previous formulas in the list, ending with φ. We write $\vdash_{\mathcal{L}}$ if such a proof exists.

Suppose that T is a set of $\mathcal{L}$-formulas. A *formal $\mathcal{L}$-proof* of φ from T is an $\mathcal{L}$-proof of $(\psi_0 \wedge \cdots \wedge \psi_n) \to \varphi$ for some $\psi_0, \ldots, \psi_n \in T$. We write $T \vdash_{\mathcal{L}} \varphi$ if such a proof exists.

Note that $\exists^{\to}$-introduction is not allowed as a rule for proofs from a set T of formulas. (A more elegant version of Hilbert's calculus, with an $\exists^{\to}$-introduction rule that is allowed in proofs from theories, would be desirable.) The advantage of this calculus (and its variants) is that the rules are simple. However, writing down actual formal proofs can be complicated and may involve many steps.

[7] This calculus is used in [2] and many other books on mathematical logic.

It will follow from the compactness theorem that $\vdash_{\mathcal{L}}$ is equivalent for differerent languages $\mathcal{L}$, so we will later write $\vdash$ instead of $\vdash_{\mathcal{L}}$.

changed from $\top := (\forall x(x \doteq x))$. The advantage that $\top$ is now a tautology.

Let $\top = (\varphi_0 \vee (\neg\varphi_0))$ (*true*) for a fixed formula φ_0 and $\bot := (\neg\top)$ (*false*).

Definition 1.5.14. An $\mathcal{L}$-theory is called

(1) *(syntactically) $\mathcal{L}$-consistent* if $T \nvdash_{\mathcal{L}} \bot$, i.e. one cannot prove a contradiction from T.
(2) *(syntactically) $\mathcal{L}$-complete* if for every $\mathcal{L}$-formula φ, $T \vdash_{\mathcal{L}} \varphi$ or $T \vdash_{\mathcal{L}} \neg\varphi$.

Proposition 1.5.15. *(Compactness for $\vdash$) An $\mathcal{L}$-theory T is $\mathcal{L}$-consistent if every finite subset of T is $\mathcal{L}$-consistent.*

Proof. By definition of $\vdash_{\mathcal{L}}$. □

Syntactic-semantic duality

	Syntactic (proof theoretic)	*Semantic (model theoretic)*
Implication	$T \vdash \varphi$	$T \models \varphi$
Consistency/Satisfiability	$T \nvdash \bot$	$T \not\models \bot$, i.e. T has a model
Completeness	For all φ, $T \vdash \varphi$ or $T \vDash \neg\varphi$	For all φ, $T \vDash \varphi$ or $T \vDash \neg\varphi$
Compactness	$T \vdash \varphi \Rightarrow$ there is a finite $T_0 \subseteq T$ with $T_0 \vdash \varphi$	$T \models \varphi \Rightarrow$ there is a finite $T_0 \subseteq T$ with $T_0 \models \varphi$

We will see in chapter 3 that the Hilbert calculus is complete, i.e. it can prove anything that can be proved by any other means. Moreover, $\vdash$ and $\vDash$ are equivalent. This will show that the left and right side in each box are equivalent.

We next give some examples how to construct formal proofs.

Example 1.5.16.

(1) ($\forall^{\to}$-axiom) Suppose that φ is an $\mathcal{L}$-formula, t is an $\mathcal{L}$-term and x is a variable. If the substitution $\varphi\frac{t}{x}$ is allowed, then the formula

$$\forall x\ \varphi \to \varphi\frac{t}{x}$$

is provable.

Proof. $\neg\varphi\frac{t}{x} \to \exists x\ \neg\varphi$ is an $^{\to}\exists$-axiom. Note that

$$(\neg\varphi\frac{t}{x} \to \exists x\ (\neg\varphi)) \ \longleftrightarrow\ (\forall x\ \varphi \to \varphi\frac{t}{x})$$

is a tautology obtained from the propositional tautology $(\neg p \to q) \leftrightarrow (\neg q \to p)$. (Recall that $\forall x\ \varphi$ is an abbreviation for $\neg\exists x\ (\neg\varphi)$.) Modus Ponens yields the required formula. □

(2) ($^{\to}\forall$-introduction) If $\varphi \to \psi$ is provable and x is not free in φ, then $\varphi \to \forall x\psi$ is provable.

Note that a special case $\varphi = (\theta \to \theta)$ (or any other tautology), $\varphi \to \psi$ is provable if and only if ψ is provable. We thus obtain the following special case of $^{\to}\forall$-introduction (for any variable x):

If ψ is provable, then $\forall x\psi$ is provable.

Proof. Note that $\neg\psi \to \neg\varphi$ is provable, since $(\varphi \to \psi) \leftrightarrow (\neg\psi \to \neg\varphi)$ is a tautology. Then $(\exists x\neg\psi) \to \neg\varphi$ holds by ($\exists^{\to}$ introduction) Using the tautology

$$((\exists x\neg\psi) \to \neg\varphi) \ \longleftrightarrow\ (\varphi \to \forall x\psi),$$

obtained from the propositional tautology $(p \to \neg q) \leftrightarrow (q \to \neg p)$, Modus Ponens yields $\varphi \to \forall x\psi$. □

Recall that $\mathcal{L}$-provability of ψ from T means that $\vdash_{\mathcal{L}} (\varphi_0 \wedge \cdots \wedge \varphi_n) \to \psi$ for some $\varphi_0, \ldots, \varphi_n \in T$. If T is a theory, then $\varphi_0, \ldots, \varphi_n$ do not contain free variables. Hence by the $\forall^{\to}$-axiom and $^{\to}\forall$-introduction, $T \vdash_{\mathcal{L}} \psi$ is equivalent to $T \vdash_{\mathcal{L}} \forall x_0, \ldots, x_k\ \psi$, if $x_0, \ldots, x_k$ are the free variables of ψ.

Lecture 6
28. April

(3) Next is a simple example of $\exists^{\to}$-introduction: to prove an existential formula, one provides a term witnessing it.

Suppose that $\mathcal{L}$ contains a unary function symbol S. Then $\forall x\ \exists y\ (S(x) = y)$ is provable.

mistakes have been corrected

Proof. $\forall x\ (S(x) \doteq S(x))$ is an equality axiom. By the $\forall^{\to}$-axiom and Modus Ponens, $S(x) \doteq S(x)$. $\exists^{\to}$-introduction yields $S(x) \doteq S(x) \to \exists y\ (S(x) = y)$. By Modus Ponens, $\exists y\ (S(x) = y)$ is provable as well. By $^{\to}\forall$-introduction, $\forall x\ \exists y\ (S(x) = y)$ is provable. □

(4) In the next example, one needs to remove the quantifier $\forall$ before applying tautologies.

The formula $(\forall x\ (\varphi \wedge \psi)) \to (\forall x\ \varphi)$ is provable.

Note that it follows by tautologies that $(\exists x\ \varphi) \to (\exists x\ (\varphi \vee \psi))$ is provable.

Proof. Note that $(\forall x\ (\varphi \wedge \psi)) \to (\varphi \wedge \psi)$ is an $\forall^{\to}$-axiom and $(\varphi \wedge \psi) \to \varphi$ is a tautology. By tautologies, $(\forall x\ (\varphi \wedge \psi)) \to \varphi$ is provable. By $^{\to}\forall$-introduction, $(\forall x\ (\varphi \wedge \psi)) \to \forall x\ \varphi$ is provable □

(5) In the next example, one has to work backwards to construct a proof. Again we leave out several steps using tautologies.

The formula $(\forall x\ (\varphi \to \psi)) \to (\exists x\ \varphi \to \exists x\ \psi)$ is provable.

Proof. By tautologies and $\exists^{\to}$-introduction, it suffices to show that

$$\varphi \to (\forall x\ (\varphi \to \psi) \to \exists x\ \varphi)$$

is provable. Again by tautologies,

$$(\varphi \wedge \forall x\ (\varphi \to \psi) \to \exists x\ \varphi)$$

suffices. Note that $\forall x\ (\varphi \to \psi) \to (\varphi \to \psi)$ is an $\forall^{\to}$-axiom and $\psi \to \exists x\ \psi$ is provable by $^{\to}\exists$-introduction. Tautologies yield the claim. □

The next lemma shows that the role of free variables in a provable formula is the same as the role of new constants in an extended language. This will be used in chapter 3 in the proof of the completeness of Hilbert's calculus.

Lemma 1.5.17. *Suppose that φ is an $\mathcal{L}$-formula, $x_0, \ldots, x_n$ are (among the) free variables in φ, C is a set of new constants and $c_0, \ldots, c_n \in C$ are distinct. Then*

$$\vdash_{\mathcal{L}\cup C} \varphi(\frac{c_0}{x_0}, \ldots, \frac{c_n}{x_n}) \Longleftrightarrow \vdash_{\mathcal{L}} \varphi.$$

Proof. Suppose that P[8] is an $\mathcal{L} \cup C$-proof of $\varphi(\frac{c_0}{x_0}, \ldots, \frac{c_n}{x_n})$, where $k \geq n$ and $c_0, \ldots, c_k$ are distinct. We choose new variables $y_0, \ldots, y_k$ that do not appear in the proof. By replacing c_i by y_i everywhere in P, we obtain a $\mathcal{L}$-proof $P\frac{y_0}{c_0}, \ldots, \frac{y_k}{c_k}$[9] of $\forall y_0, \ldots, y_n\ \varphi(\frac{y_0}{x_0}, \ldots, \frac{y_n}{x_n})$. (One can easily check that each axiom and rule remains valid.) By the $\forall^{\to}$-axiom $\forall y_0, \ldots, y_n\ \varphi(\frac{y_0}{x_0}, \ldots, \frac{y_n}{x_n}) \to \varphi$ (the x_i are not free in the formula on the left). By Modus Ponens, we obtain $\vdash_{\mathcal{L}} \varphi$.

[8]A proof is a finite sequence of formulas.

[9]Up to now, we only defined substitution of variables by terms in formulas. If we were more precise here, we would define substitution of constants by variables in formulas, and thus in proofs, in precisely the same way.

Conversely, suppose that $\vdash_{\mathcal{L}} \varphi$ holds. By ${}^{\rightarrow}\forall$-introduction, we have $\vdash_{\mathcal{L}} \forall x_0, \ldots, x_n\ \varphi$. By the $\forall^{\rightarrow}$-axiom and Modus Ponens, $\vdash_{\mathcal{L}\cup C} \varphi(\frac{c_0}{x_0}, \ldots, \frac{c_n}{x_n})$. □

Note that the special case of the previous lemma where one chooses no variables at all shows that $\vdash_{\mathcal{L}} \varphi \Longleftrightarrow \vdash_{\mathcal{L}\cup C} \varphi$ holds for any $\mathcal{L}$-formula φ and a set C of constants. Hence the meaning of $\vdash_{\mathcal{L}}$ does not change when $\mathcal{L}$ is enriched by constants. will later see that this is also true for relation and function symbols.

Ending this chapter, we have a look at Hilbert's program, as paraphrased in [Kossak: Mathematical Logic (2018), page 180]:

(1) "Define a system based on a formal language in which all mathematical statements can be expressed, and in which proofs of theorems can be carried out according to well-defined, strict rules of proof.
(2) Show that the system is *complete*, i.e. all true mathematical statement can be proved in the formalism.
(3) Show that the system is *consistent*, i.e. it is not possible to derive a statement and its negation. The consistency should be carried out using finitistic means without appeal to the notion of actual infinity.
(4) Show that the system is *conservative*, i.e. if a statement about concrete objects of mathematics, such as natural numbers of geometric figures, has a proof involving infinitistic methods, then is also has an elementary proof in which those methods are not used.
(5) Show that the system is *decidable* by finding an algorithm for deciding the truth of falsity of any mathematical statement."

We completed (1) in this chapter, and (2) is the completeness of Hilbert's calculus proved in chapter 3.

But the other items cannot be realised: (5) is false by (the proof of) Gödel's first incompleteness theorem; (3) and (4) are false by Gödel's second incompleteness theorem. The failure of (4) also follows from the unprovability in PA of the convergence of Goodstein sequences.

References

[1] Lorenz Halbeisen and Regula Krapf. Gödel's Theorems & Zermelo's Axioms.
[2] Martin Ziegler. *Mathematische Logik*. Springer, 2010.