Models of Set Theory I. - Summer 2019

PD Dr. Philipp Lücke Problem sheet 3

Problem 9 (4 Points)**.** Examine which axioms of ZFC hold in the class $\mathrm{V} \setminus \mathrm{V}_\omega$.

Problem 10 (3 points)**.** Given a singular cardinal κ, examine which axioms of ZFC hold in $H(\kappa)$.

Problem 11. Let $\mathcal{L}_*$ denote the first-order language that extends $\mathcal{L}_\in$ by a unary predicate symbol $\dot{C}$ and let ZFC_* denote the canonical $\mathcal{L}_*$-theory that extends ZFC by expanding the Replacement and Separation schema to ZFC_*-formulas. Given an $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_{n-1})$, we let Refl_φ denote the $\mathcal{L}_*$-sentence

$$\forall \alpha \in \mathrm{Ord} \ \big[\alpha \in \dot{C} \ \longrightarrow \ \forall z_0, \ldots, z_{n-1} \in \mathrm{V}_\alpha \ [\varphi(z_0, \ldots, z_{n-1}) \longleftrightarrow \varphi^{\mathrm{V}_\alpha}(z_0, \ldots, z_{n-1})]\big].$$

(1) (6 points) Show that the theory

$$T \ = \ \mathsf{ZFC}_* \ + \ \{\mathsf{Refl}_\varphi \mid \varphi \textit{ is an } \mathcal{L}_\in\textit{-formula}\} \ + \ \text{"}\dot{C} \textit{ is a closed unbounded class of cardinals}\text{"}$$

is consistent relative to ZFC.

(2) (1 points) Show that

$$T \vdash \forall \alpha \ [\alpha \in \dot{C} \ \longrightarrow \ \text{"}\alpha \textit{ is a strong limit cardinal}\text{"}].$$

Problem 12 (6 points)**.** Complete the proof of the Σ-*Recursion Theorem*: Given $0 < n < \omega$, an $\mathcal{L}_\in$-formula $\psi(v_0, \ldots, v_{n+1})$ and Σ_n-formulas $\varphi_0(v_0, \ldots, v_{n+2})$ and $\varphi(v_0, \ldots, v_{n+1})$, there is a Σ_n-formula $\Phi(v_0, \ldots, v_{n+1})$ such that the theory $\mathsf{ZF}^- - (\textit{Infinity})$ proves the following $\mathcal{L}_\in$-sentence:

For all $z_0, \ldots, z_{n-1}$, if

$$R \ = \ \{\langle a, b\rangle \mid \psi(a, b, z_0, \ldots, z_{n-1})\}$$

is a strongly well-founded relation,

$$G \ = \ \{\langle\langle a_0, a_1\rangle, b\rangle \mid \varphi_0(a_0, a_1, b, z_0, \ldots, z_{n-1})\}$$

is a class function with domain $\mathrm{V} \times \mathrm{V}$ and

$$P \ = \ \{\langle a, b\rangle \mid \varphi(a, b, z_0, \ldots, z_{n-1})\}$$

is a class function with domain V and $P(a) = \{b \mid bRa\}$ for all sets a, then

$$F = \{\langle a, b\rangle \mid \Phi(a, b, , z_0, \ldots, z_{n-1})\}$$

is a class function with domain V and $F(a) = G(a, F \restriction P(a))$ for all sets a".

Please hand in your solutions on Tuesday, April 23, before 10am.