Models of Set Theory I. - Summer 2019

PD Dr. Philipp Lücke Problem sheet 2

Problem 5 (4 Points)**.** Prove Lemma 1.1.6.(ii) from the lecture course: Assume ZF^-. Let W be a non-empty transitive class and $\varphi(v_0, \ldots, v_{n+1})$ be an $\mathcal{L}_\in$-formula. Then $(\text{Replacement}_\varphi)^W$ holds if and only if for all $c_0, \ldots, c_{n-1} \in W$, either there are $a, b_0, b_1 \in W$ with $\varphi^W(a, b_i, c_0, \ldots, c_{n-1})$ for all $i < 2$, or there is an $a \in W$ with $\neg\varphi^W(a, b, c_0, \ldots, c_{n-1})$ for all $b \in W$, or

$$\{b \in W \mid \exists a \in d \; \varphi^W(a, b, c_0, \ldots, c_{n-1})\} \in W$$

for every $d \in W$.

Problem 6 (4 points)**.** Show that the following statements are equivalent for every $\mathcal{L}_\in$-theory T extending ZF and every $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_{n-1})$:

(1) φ is a Σ_1^{T}-formula.

(2) There is a $\mathcal{L}_\in$-formula $\psi(v_0, \ldots, v_{n-1})$ with

$$\mathsf{T} \vdash \forall x_0, \ldots, x_{n-1} \; \big[\varphi(x_0, \ldots, x_{n-1}) \longleftrightarrow \exists z \; \big(\text{"}z \textit{ is transitive}\text{"} \wedge x_0, \ldots, x_{n-1} \in z \wedge \psi^z(x_0, \ldots, x_{n-1})\big)\big].$$

Problem 7 (4 points)**.** Let $\varphi(v)$ be the canonical $\mathcal{L}_\in$-formula stating that

"*v is a strongly inaccessible cardinal*".

Given a strongly inaccessible cardinal κ, show that φ is V_κ-absolute.

Problem 8 (8 points)**.** Prove Theorem 1.1.17. from the lecture course: Let κ be an uncountable regular cardinal.

(1) $(\mathsf{ZFC}^-)^{H(\kappa)}$.

(2) $(\text{Collection}_\varphi)^{H(\kappa)}$ for every $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_{n+1})$.

(3) The following statements are equivalent:

   (a) $\mathsf{ZFC}^{H(\kappa)}$.

   (b) $H(\kappa) = V_\kappa$.

   (c) κ is strongly inaccessible.

Please hand in your solutions on Monday, April 15, before the lecture.