Set Theory - Winter 2018/19

Prof. Dr. Peter Koepke, PD Dr. Philipp Lücke Problem sheet 12

Problem 46 (4 points)**.** Prove the following statement: if κ is the minimal cardinal that carries a non-trivial 2-valued measure, then κ is a measurable cardinal.

Problem 47 (4 points)**.** Prove that an infinite ordinal α is countable if and only if there exists a measure μ on a set X and a sequence $\langle X_i \mid i < \alpha \rangle$ of disjoint subsets of X such that $\mu(X_i) > 0$ holds for all $i < \alpha$.

Problem 48. Fix a sequence $\langle f_\beta : \omega \longrightarrow \beta \mid \beta < \aleph_1 \rangle$ of surjections. Given $\alpha < \aleph_1$ and $n < \omega$, define

$$A_{\alpha,n} \;=\; \{\beta < \aleph_1 \mid f_\beta(n) = \alpha\}.$$

Prove the following statements:

(1) (1 point) If $\alpha < \beta < \aleph_1$ and $n < \omega$, then $A_{\alpha,n} \cap A_{\beta,n} = \emptyset$.

(2) (1 point) If $\alpha < \aleph_1$, then $\aleph_1 \setminus \bigcup\{A_{\alpha,n} \mid n < \omega\} \subseteq \alpha + 1$.

(3) (2 points) $\aleph_1$ does not carry a non-trivial measure.

Problem 49. Assume that μ is a non-trivial measure on a set X such that $\mu(X) = 1$ and for every subset Y of X with $\mu(Y) > 0$, there are disjoint subsets Y_0 and Y_1 of Y with $\mu(Y_0) > 0$ and $\mu(Y_1) > 0$.

(1) (4 points) Show that for every subset Y of X and every $\epsilon > 0$, there is an $n < \omega$ and a partition $\langle Y_i \mid i < n \rangle$ of Y such that $\mu(Y_i) < \epsilon$ for all $i < n$.

(2) (2 points) Show that for every $r \in [0,1]$ and every $\epsilon > 0$, there is a subset Y of X with $|\mu(Y) - r| < \epsilon$.

(3) (2 points) Show that for every $r \in [0,1]$, there is a subset Y of X with $\mu(Y) = r$.

Please hand in your solutions on Wednesday, January 16 before the lecture (Briefkästen 6 & 7).