Set Theory - Winter 2018/19

Prof. Dr. Peter Koepke, PD Dr. Philipp Lücke Problem sheet 11

Problem 42 (4 points)**.** Assume that $2^\lambda = \lambda^+$ holds for every singular cardinal λ with $2^{\operatorname{cof}(\lambda)} < \lambda$ and determine the value of 2^κ from the value of $2^{<\kappa}$ for all singular cardinals κ.

Problem 43 (4 points)**.** Let S be a stationary subset of a regular uncountable cardinal κ. A subset C of κ is an *S-cub* if it is unbounded in κ and $\sup(x) \in C$ holds for every $x \subseteq C$ with $\sup(x) \in S$. Given a sequence $(C_\alpha \mid \alpha < \lambda)$ of S-cubs with $\lambda < \kappa$, show that $\bigcap_{\alpha<\lambda} C_\alpha$ is an S-cub.

Problem 44 (8 points)**.** Let S be a stationary subset of a regular uncountable cardinal λ. Given an ordinal α, let c^λ_α denote the constant function with domain λ and range $\{\alpha\}$.

(1) Prove that $\|c^\lambda_\alpha\|_S \geq \alpha$ holds for all $\alpha \in \mathrm{Ord}$.
(2) Determine the value of $\|c^\lambda_\alpha\|_S$ for all $\alpha < \lambda$.
(3) Prove that $\|c^\lambda_\lambda\|_S > \lambda$ (Hint: *Use the identity function of* λ).
(4) Prove that there is a proper class of ordinals α with $\|c^\lambda_\alpha\|_S = \alpha$.

Problem 45 (4 points)**.** Let κ be an uncountable regular cardinal and let $\lambda \geq \kappa$ be a cardinal. Define

$$\mathcal{P}_\kappa(\lambda) \ = \ \{a \subseteq \lambda \mid \operatorname{card}(a) < \kappa\}.$$

A subset C of $\mathcal{P}_\kappa(\lambda)$ is *closed unbounded in* $\mathcal{P}_\kappa(\lambda)$ if the following statements hold:

(a) For all $a \in \mathcal{P}_\kappa(\lambda)$, there is $c \in C$ with $a \subseteq c$.
(b) If $(c_\alpha \mid \alpha < \lambda)$ is a sequence of elements of C with $\lambda < \kappa$ and $c_\alpha \subseteq c_\beta$ for all $\alpha \leq \beta < \lambda$, then $\bigcup_{\alpha<\lambda} c_\alpha$ is an element of C.

Moreover, a subset S of $\mathcal{P}_\kappa(\lambda)$ is *stationary in* $\mathcal{P}_\kappa(\lambda)$ if $C \cap S \neq \emptyset$ holds for every closed unbounded subset C of $\mathcal{P}_\kappa(\lambda)$. Finally, a function $r : D \longrightarrow \lambda$ with $D \subseteq \mathcal{P}_\kappa(\lambda)$ is *regressive* if $r(a) \in a$ holds for all $a \in D$. Prove that, if S is a stationary subset of $\mathcal{P}_\kappa(\lambda)$ and $r : S \longrightarrow \lambda$ is regressive, then there is $E \subseteq S$

stationary in $\mathcal{P}_\kappa(\lambda)$ with the property that $r \upharpoonright E$ is constant (Hint: *Given a sequence* $(C_\gamma \mid \gamma < \lambda))$ *of closed unbounded subsets of* $\mathcal{P}_\kappa(\lambda)$, *consider the set*

$$\mathop{\triangle}_{\gamma<\lambda} C_\gamma \;=\; \{a \in \mathcal{P}_\kappa(\lambda) \mid a \in \bigcap_{\gamma \in a} C_\gamma\}$$

and imitate the proof of Fodor's Lemma).

Please hand in your solutions on Wednesday, January 09 before the lecture (Briefkästen 6 & 7).

– *Happy holidays!* –