Set Theory - Winter 2018/19

Prof. Dr. Peter Koepke, PD Dr. Philipp Lücke — Problem sheet 10

Problem 38 (4 points)**.** Prove that the following statements are equivalent for all limit ordinals $\lambda \leq \kappa$:

(1) $\mathrm{cof}(\kappa) = \lambda$.

(2) λ is the maximal regular cardinal with the property that there exists a cofinal and strictly increasing function $c : \lambda \longrightarrow \kappa$.

(3) λ is the minimal limit ordinal with the property that there exists a cofinal, strictly increasing and continuous function $c : \lambda \longrightarrow \kappa$.

Problem 39 (6 points)**.** A sequence $(A_\alpha \mid \alpha \in \mathrm{Lim} \cap \aleph_1)$ is a *ladder system* if A_α is a cofinal subset of α of order-type ω for every countable limit ordinal α. Derive the following statements from the axioms of ZF :

(1) If there exists a ladder system, then there exists a sequence

$$(s_\alpha : \omega \longrightarrow \alpha \mid \omega \leq \alpha < \aleph_1)$$

of surjections (Hint: *Construct the sequence with the help of the* Recursion Theorem).

(2) If every countable set has a choice function und there exists a ladder system, then there exists a stationary subset B of $\aleph_1$ with the property that $\aleph_1 \setminus B$ is also stationary in $\aleph_1$ (Hint: *Use ideas from the proof of Theorem 148*).

Problem 40 (6 points)**.** Let $\omega < \lambda = \mathrm{cof}(\kappa) < \kappa \in \mathrm{Card}$ with $\mu^\lambda < \kappa$ for all $\mu < \kappa$ and let $(\kappa_\alpha < \kappa \mid \alpha < \lambda)$ be a sequence that is cofinal in κ and continuous. Prove the following statements:

(1) If $\mathcal{F} \subseteq \prod_{\alpha<\lambda} A_\alpha$ is almost disjoint with $\mathrm{card}(A_\alpha) \leq \kappa_\alpha^{++}$ for all $\alpha < \lambda$, then $\mathrm{card}(\mathcal{F}) \leq \kappa^{++}$.

(2) If $2^\mu = \mu^{++}$ for all $\mu \in \kappa \cap \mathrm{Card}$, then $2^\kappa \leq \kappa^{++}$.

(Hint: *Use the methods of the proof of* Silvers Theorem)

Problem 41 (4 points)**.** A linear order $(L, \lhd)$ is $\aleph_1$-like if

$$\operatorname{card}(\{a \in L \mid a \lhd b\}) < \aleph_1$$

holds for all $b \in L$. Prove the following statements:

(1) If $(L, \lhd)$ is an $\aleph_1$-like linear order, then $\operatorname{card}(L) \leq \aleph_1$.

(2) There is an $\aleph_1$-like linear order that is dense and uncountable.

Please hand in your solutions on Wednesday, December 19 before the lecture (Briefkästen 6 & 7).