Set Theory - Winter 2018/19

Prof. Dr. Peter Koepke, PD Dr. Philipp Lücke Problem sheet 9

Problem 33 (4 points)**.** Prove Lemma 133 from the lecture course: if $\kappa \in \mathrm{Card}$ with $\mathrm{cof}(\kappa) > \omega$ and $(C_i \mid i < \gamma)$ is a sequence of closed unbounded subsets of κ with $\gamma < \mathrm{cof}(\kappa)$, then $\bigcap_{i<\gamma} C_i$ is closed unbounded in κ.

Problem 34 (6 points)**.** Let κ be an uncountable regular cardinal. For every function $f : \kappa \longrightarrow \kappa$, the set of *closure points of* f is defined as

$$C_f \;=\; \{\alpha < \kappa \mid \alpha > 0,\; f[\alpha] \subseteq \alpha\}.$$

Prove the following statements:

(1) For every function $f : \kappa \longrightarrow \kappa$, the set C_f is closed unbounded in κ.

(2) If C is a closed unbounded subset of κ, then there exists a function $f : \kappa \longrightarrow \kappa$ with $C_f \subseteq C$.

(3) If $f : \kappa \longrightarrow \kappa$ is strictly increasing and continuous (i.e. $f(\lambda) = \bigcup_{\alpha<\lambda} f(\alpha)$ holds for every limit ordinal $\lambda < \kappa$), then the set

$$\mathrm{Fix}(f) \;=\; \{\alpha < \kappa \mid f(\alpha) = \alpha\}$$

of *fixed points of* f is closed unbounded in κ.

Problem 35 (4 points)**.** Let κ be an uncountable regular cardinal. Prove that the following statements are equivalent for every subset X of κ:

(1) X is stationary in κ.

(2) For every regressive function $f : S \longrightarrow \kappa$, there is an $\alpha < \kappa$ with the property that the set $f^{-1}[\{\alpha\}]$ is stationary in κ.

Problem 36 (6 points)**.** Remember that, given $\beta \in \mathrm{Ord}$, we let ${}^{\beta}2$ denote the set of all functions from β to 2. Moreover, we write ${}^{<\beta}2 = \bigcup_{\alpha<\beta} {}^{\alpha}2$. Finally, if T is a subset of ${}^{<\beta}2$, then we define

$$[T]_\beta \;=\; \{f \in {}^{\beta}2 \mid \forall \alpha < \beta \; f \restriction \alpha \in T\}.$$

Given infinite cardinals $\nu < \kappa$ with κ regular and $\nu^+ < \kappa$, prove that $[T]_\kappa \neq \emptyset$ holds for all $T \subseteq {}^{<\kappa}2$ satisfying the following statements

(1) If $f \in T$ and $\alpha \in \mathrm{dom}(f)$, then $f \restriction \alpha \in T$.

(2) If $\alpha < \kappa$, then $0 < \mathrm{card}(T \cap {}^{\alpha}2) \leq \nu$.

(Hint: *Show that for every* $\delta \in E^{\kappa}_{\nu^+}$*, there is a* $\gamma < \delta$ *with the property that* $f \restriction \gamma \neq g \restriction \gamma$ *holds for all* $f, g \in T \cap {}^{\delta}2$ *with* $f \neq g$).

Please hand in your solutions on Wednesday, December 12 before the lecture (Briefkästen 6 & 7).