Set Theory - Winter 2018/19

Prof. Dr. Peter Koepke, PD Dr. Philipp Lücke — Problem sheet 8

Problem 29. Prove the following statements:

(1) (1 point) A regular cardinal is weakly inaccessible if and only if it is an uncountable limit cardinal.

(2) (1 point) A regular cardinal κ is strongly inaccessible if and only if it is uncountable and $2^\nu < \kappa$ holds for all cardinals $\nu < \kappa$.

(3) (2 points) A regular cardinal κ is strongly inaccessible if and only if $\mathrm{card}(V_\kappa) = \kappa > \aleph_0$.

Problem 30 (4 points)**.** Prove that for all $\alpha, \beta \in \mathrm{Lim}$, all $\gamma \in \mathrm{Ord}$ with $\aleph_\gamma < \mathrm{cof}(\beta)$ and every sequence $(\alpha_\xi \mid \xi < \beta)$ that is strictly increasing and cofinal in α, we have

$$\aleph_\alpha^{\aleph_\gamma} = \sum_{\xi<\beta} \aleph_{\alpha_\xi}^{\aleph_\gamma} = \bigcup_{\xi<\beta} \aleph_{\alpha_\xi}^{\aleph_\gamma}.$$

Problem 31 (6 points)**.** Prove the following statements:

(1) If $2^{\aleph_1} < \aleph_\omega$ and $\gimel(\aleph_\omega) > \aleph_{\aleph_1}$, then $\gimel(\aleph_{\aleph_1}) = \gimel(\aleph_\omega)$.

(2) If $2^{\aleph_0} \geq \aleph_{\aleph_1}$, then $\gimel(\aleph_\omega) = 2^{\aleph_0}$ and $\gimel(\aleph_{\aleph_1}) = 2^{\aleph_1}$.

(3) If $2^{\aleph_0} = \aleph_{\omega+1}$ and $2^{\aleph_1} = \aleph_{\omega+2}$, then $\aleph_\omega^{\aleph_0} < 2^{\aleph_\omega}$. In particular, if $2^{\aleph_0} = \aleph_{\omega+1}$, $2^{\aleph_1} = \aleph_{\omega+2}$ and $2^{\aleph_2} = 2^{\aleph_\omega}$, then the $\gimel$-function is not monotone.

Problem 32 (6 points)**.** Prove the following statements:

(1) Given $m < \omega$, we have $2^{\aleph_0} \leq \aleph_m$ if and only if $\aleph_n^{\aleph_0} = \aleph_n$ holds for all $m \leq n < \omega$.

(2) For all $\kappa \in \mathrm{Card}$, we have $2^\kappa = (2^{<\kappa})^{\mathrm{cof}(\kappa)}$.

(3) If $\kappa \in \mathrm{Card}$ with $2^\nu < \kappa$ for all cardinals $\nu < \kappa$, then $2^\kappa = \kappa^{\mathrm{cof}(\kappa)}$.

Please hand in your solutions on Wednesday, December 05 before the lecture (Briefkästen 6 & 7).