Set Theory - Winter 2018/19

Prof. Dr. Peter Koepke, PD Dr. Philipp Lücke — Problem sheet 7

Problem 25 (4 points)**.** Prove that the relation $<^2$ on $\mathrm{Ord} \times \mathrm{Ord}$ is a strong well-order.

Problem 26 (6 points)**.** Let $G : \mathrm{Ord} \times \mathrm{Ord} \longrightarrow \mathrm{Ord}$ denote the *Gödel pairing function*.

(1) Prove that $G[\alpha \times \alpha] \subseteq \omega^\alpha$ holds for all $\alpha \in \mathrm{Ord}$ (where ω^α denotes ordinal exponentiation).

(2) Prove that for every $\omega \leq \alpha \in \mathrm{Ord}$, there is $\beta \in \mathrm{Ord}$ satisfying $\alpha < \beta$, $\mathrm{card}(\alpha) = \mathrm{card}(\beta)$ and $G[\beta \times \beta] = \beta$.

Problem 27 (6 points)**.** Given $\gamma, \delta \in \mathrm{Ord}$, define $\mathcal{FIN}(\gamma, \delta)$ to be the set of all functions $f : \gamma \longrightarrow \delta$ with $\{\beta < \gamma \mid f(\beta) \neq 0\}$ finite and let $\lhd$ be the relation on $\mathcal{FIN}(\gamma, \delta)$ defined by

$$f \lhd g \iff f \neq g \wedge \exists \alpha < \gamma \, [f(\alpha) < g(\alpha) \wedge \forall \beta < \gamma \, (\beta > \alpha \longrightarrow f(\beta) = g(\beta))].$$

(1) Prove that for all $\gamma, \delta \in \mathrm{Ord}$, the relation $\lhd$ is a well-ordering of $\mathcal{FIN}(\gamma, \delta)$.

(2) Determine the order-type of $(\mathcal{FIN}(\omega, 2), \lhd)$.

Problem 28 (4 points)**.** Given an ordinal $\lambda > 0$, we let τ_λ denote the order topology on λ induced by $<$, i.e. the topology whose basic open sets are the intervals

$$(\beta, \gamma) = \{\alpha \in \mathrm{Ord} \mid \beta < \alpha < \gamma\}$$

for $\beta < \gamma \leq \lambda$ and

$$[0, \beta) = \{\alpha \in \mathrm{Ord} \mid \alpha < \beta\}$$

for $\beta \leq \lambda$.

Prove that the following statements hold for all ordinals $\lambda > 0$:

(1) A subset $x \subseteq \lambda$ is closed in τ_λ if and only if $\bigcup y \in x$ holds for all $y \subseteq x$ with $\bigcup y \in \lambda$.

(2) The set λ is compact in τ_λ if and only if λ is a successor ordinal.

Please hand in your solutions on Wednesday, November 28 before the lecture (Briefkästen 6 & 7).

On December 3rd, 7 pm (s.t.), in the Lipschitz hall the Fachschaft (student council) organizes a Ladies Night for all female students enrolled in the Bachelor's program for mathematics (5th semester or higher) or in the Master's program. In case of questions, you can contact us via `gleichstellung@fsmath.uni-bonn.de`.