Set Theory - Winter 2018/19

Prof. Dr. Peter Koepke, PD Dr. Philipp Lücke — Problem sheet 6

Problem 22 (8 points)**.** The *Axiom of Dependent Choice* DC is the following statement:

> "*If R is a binary relation on a set a with* $\mathrm{dom}(R) = a$ *and $x \in a$, then there is a function $f : \omega \longrightarrow a$ with $f(0) = x$ and $f(n)\ R\ f(n+1)$ for all $n < \omega$.*"

Derive the following statements from the axioms of ZF:

(1) The *Axiom of Choice* AC implies DC.

(2) If DC holds and $\lhd$ is a binary relation on a set b, then the following statements are equivalent:

 (a) The relation $\lhd$ is well-founded.

 (b) There is no function $f : \omega \longrightarrow b$ with $f(n+1) \lhd f(n)$ for all $n < \omega$.

(3) DC implies that every countable set has a choice function.

(4) If every countable set has a choice function and $f : \omega \longrightarrow \aleph_1$, then there is an $\alpha < \aleph_1$ with $\mathrm{ran}(f) \subseteq \alpha$.

Problem 23. Derive the following statements from the axioms of ZF:

(1) (2 points) If z is a set and $h : \mathcal{P}(z) \longrightarrow \mathcal{P}(z)$ is a function satisfying $h(u) \subseteq h(v)$ for all $u \subseteq v \subseteq z$, then there is $u \subseteq z$ with $h(u) = u$ (Hint: *Consider the set* $\{u \subseteq z \mid u \subseteq h(u)\}$).

(2) (2 points) If $x \preccurlyeq y$ and $y \preccurlyeq x$, then $x \sim y$ (Hint: *Given injections $f : x \longrightarrow y$ and $g : y \longrightarrow x$, consider the function*

$$h : \mathcal{P}(x) \longrightarrow \mathcal{P}(x); u \mapsto x \setminus g[y \setminus f[u]]$$

and use (1)).

(3) (4 points) For every cardinal κ, there exists a cardinal η with $\kappa \prec \eta$ (Hint: *Consider the set* $\{R \mid R$ *is a well-ordering of a subset of* $\kappa\}$ *and the class* $\{\alpha \in \mathrm{Ord} \mid \alpha \preccurlyeq \kappa\}$. *Then use the Mostowski-collapse and the Replacement scheme*).

Problem 24 (4 points)**.** Show that the axioms of ZFC imply the existence of a cardinal κ with $\aleph_\kappa = \kappa$.

Please hand in your solutions on Wednesday, November 21 before the lecture (Briefkästen 6 & 7).