Set Theory - Winter 2018/19

Prof. Dr. Peter Koepke, PD Dr. Philipp Lücke Problem sheet 4

Problem 14 (*Transitive closure*, 4 points)**.** Show that there is a unique class function $T : V \longrightarrow V$ such that the following statements hold for all sets x:

(1) $T(x)$ is transitive with $x \in T(x)$.

(2) If y is a transitive set with $x \in y$, then $T(x) \subseteq y$.

(Hint: *Use Problem 10*).

Problem 15 (*Scott's trick*, 4 points)**.** Let E be a binary relation. Show that E is an equivalence relation on V if and only if there is a class function $F : V \longrightarrow V$ with $E = \{(x, y) \mid F(x) = F(y)\}$ (Hint: *For the forward implication, use the* von Neumann Hierarchy *as well as the corresponding rank function R and, given a set x, consider the class* $\{y \mid E(x, y) \wedge \forall z\ [E(x, z) \longrightarrow R(y) \leq R(z)]\}$).

Problem 16 (4 points)**.** Define

- $I_0 = \{0\}$.
- $I_1 = \{\alpha + 1 \mid \alpha \in \mathrm{Ord}\}$.
- $I_2 = \mathrm{Lim}$.

Prove the following statements:

(1) There is a subset A of 3×3 with

$$\{(\alpha, \beta) \in \mathrm{Ord} \times \mathrm{Ord} \mid \alpha + \beta \in \mathrm{Lim}\} = \bigcup \{I_i \times I_j \mid (i, j) \in A\}.$$

(2) There is a subset B of 3×3 with

$$\{(\alpha, \beta) \in \mathrm{Ord} \times \mathrm{Ord} \mid \alpha \cdot \beta \in \mathrm{Lim}\} = \bigcup \{I_i \times I_j \mid (i, j) \in B\}.$$

Problem 17 (4 points)**.** Show that $(\mathbb{R}^+, \cdot^{\mathbb{R}}, 1^{\mathbb{R}})$ is a multiplicative group.

Problem 18 (4 points)**.** Formalize the structure $(\mathbb{C}, +^{\mathbb{C}}, \cdot^{\mathbb{C}}, 0^{\mathbb{C}}, 1^{\mathbb{C}})$ of complex numbers such that $\mathbb{R} \subseteq \mathbb{C}$, $0^{\mathbb{C}} = 0^{\mathbb{R}}$, $1^{\mathbb{C}} = 1^{\mathbb{R}}$, $+^{\mathbb{C}} \upharpoonright (\mathbb{R} \times \mathbb{R}) = +^{\mathbb{R}}$ and $\cdot^{\mathbb{C}} \upharpoonright (\mathbb{R} \times \mathbb{R}) = \cdot^{\mathbb{R}}$.

Please hand in your solutions on Wednesday, November 07 before the lecture (Briefkästen 6 & 7).