Set Theory - Winter 2018/19

Prof. Dr. Peter Koepke, PD Dr. Philipp Lücke Problem sheet 3

Problem 10 (4 points)**.** Prove the following statements:

(1) If $G : V \longrightarrow V$ is a class function, then there is a unique class function $F : V \longrightarrow V$ that satisfies $F(x) = G(F \restriction x)$ for ever set x.

(2) Given a class function G, there is a unique class function F satisfying the following statements:
   (a) Either $\mathrm{dom}(F) = \mathrm{Ord}$ or $\mathrm{dom}(F) \in \mathrm{Ord}$.
   (b) $\mathrm{dom}(F) = \{\alpha \in \mathrm{Ord} \mid F \restriction \alpha \in \mathrm{dom}(G)\}$.
   (c) $F(\alpha) = G(F \restriction \alpha)$ for all $\alpha \in \mathrm{dom}(F)$.

Problem 11. Prove the following statements:

(1) (1 point) There is a unique class function $H : \mathrm{Ord} \longrightarrow V$ satisfying the following statements:
   (a) $H(0) = \emptyset$.
   (b) $H(\alpha + 1) = \mathcal{P}(H(\alpha))$ for all $\alpha \in \mathrm{Ord}$.
   (c) $H(\alpha) = \bigcup\{H(\beta) \mid \beta < \alpha\}$ for all $\alpha \in \mathrm{Lim}$.

(2) (1 point) $V = \bigcup \mathrm{ran}(H)$.

(3) (2 points) Every set of ordinals x has a *least upper bound* $\mathrm{lub}(x)$ in the class Ord with respect to the natural ordering $<$ of Ord.

(4) (1 points) There is a unique class function $R : V \longrightarrow \mathrm{Ord}$ with

$$R(x) \;=\; \mathrm{lub}(\{R(y) \mid y \in x\})$$

for all $x \in V$ (Hint: *Use Problem 10*).

(5) (2 points) For every set x, we have $R(x) = \min\{\alpha \in \mathrm{Ord} \mid x \subseteq H(\alpha)\}$.

(6) (1 point) For every set x, the ordinal $R(x)$ is the unique element α of Ord with $x \in H(\alpha + 1) \setminus H(\alpha)$.

(7) (1 point) $R(\alpha) = \alpha$ for all $\alpha \in \mathrm{Ord}$.

Problem 12 (3 points)**.** The *Collection scheme* states that

$$\forall x \; \exists y \; \forall u \in x \; [\exists v \; R(u, v) \longrightarrow \exists w \in y \; R(u, w)]$$

holds for every binary relation R. Prove that the axioms of ZF imply the Collection scheme (Hint: *Use the function H constructed in Problem 11*).

Problem 13 (4 points)**.** Fix ordinals α and β.

(1) Let $\lhd$ denote the binary relation on the set

$$\alpha \sqcup \beta \;=\; (\alpha \times \{0\}) \;\cup\; (\beta \times \{1\})$$

with

$$(\gamma, i) \lhd (\delta, j) \;\longleftrightarrow\; [i < j \;\vee\; (i = j \wedge \gamma < \delta)]$$

for all $(\gamma, i), (\delta, j) \in \alpha \sqcup \beta$. Construct a bijection $b : \alpha + \beta \longrightarrow \alpha \sqcup \beta$ with

$$\gamma < \delta \;\longleftrightarrow b(\gamma) \lhd b(\delta)$$

for all $\gamma, \delta < \alpha + \beta$.

(2) Let $\blacktriangleleft$ denote the binary relation on $\alpha \times \beta$ with

$$(\gamma, \mu) \blacktriangleleft (\delta, \nu) \;\longleftrightarrow\; [\mu < \nu \;\vee\; (\mu = \nu \wedge \gamma < \delta)]$$

for all $(\gamma, \mu), (\delta, \nu) \in \alpha \times \beta$. Construct a bijection $b : \alpha \cdot \beta \longrightarrow \alpha \times \beta$ with

$$\gamma < \delta \;\longleftrightarrow b(\gamma) \blacktriangleleft b(\delta)$$

for all $\gamma, \delta < \alpha \cdot \beta$.

Please hand in your solutions on Wednesday, October 31 before the lecture (Briefkästen 6 & 7).