Set Theory - Winter 2018/19

Prof. Dr. Peter Koepke, PD Dr. Philipp Lücke Problem sheet 2

Problem 5 (4 points)**.** Suppose that F and G are functions.

(1) Show that $F = G$ if and only if $dom(F) = dom(G)$ and $F(x) = G(x)$ holds for all $x \in dom(F)$.

(2) Show that F is injective if and only if there is a function H with $dom(H) = ran(F)$ and $H(F(x)) = x$ for all $x \in dom(F)$.

Problem 6. Prove the following statements:

(1) (1 point) The class $\{\{x\} \mid x \in V\}$ of all *singletons* is a proper class.

(2) (1 point) A class A is a proper class if and only if the class $\{x \mid x \subseteq A\}$ is a proper class.

(3) (2 points) Construct a class A with the property that the corresponding class $\{x \in A \mid \forall y \in x \; y \notin A\}$ is a proper class.

Problem 7. Define a binary relation $\sim$ on V by setting

$$x \sim y \;\longleftrightarrow\; \exists f \; f : x \leftrightarrow y.$$

(1) (1 point) Prove that $\sim$ is an equivalence relation on V.

(2) (2 points) Determine the $\sim$-equivalence classes of $\emptyset$ and $\{\emptyset\}$.

(3) (2 points) Show that the collection of all x in V with the property that the $\sim$-equivalence class of x is a proper class is a class and that this class is proper.

Problem 8.

(1) (1 point) Show that the power set of a transitive set is transitive.

(2) (2 points) Determine the class of all ordinals α with the property that $\mathcal{P}(\alpha)$ is an ordinal.

(3) (1 point) Prove that there are transitive sets that are not ordinals.

Problem 9 (3 points)**.** The *Collection scheme* states that

$$\forall x \; \exists y \; \forall u \in x \; [\exists v \; R(u, v) \longrightarrow \exists w \in y \; R(u, w)]$$

holds for every binary relation R. Prove that the axioms and schemes of ZF without the Replacement scheme, but with the Collection scheme, imply the Replacement scheme.

Please hand in your solutions on Wednesday, October 24 before the lecture (Briefkästen 6 & 7).