Set Theory - Winter 2018/19

Prof. Dr. Peter Koepke, PD Dr. Philipp Lücke — Problem sheet 1

Problem 1 (4 points)**.** Prove the following statements:

(1) $\bigcup V = V$.
(2) $\bigcap V = \emptyset$.
(3) $\bigcup \emptyset = \emptyset$.
(4) $\bigcap \emptyset = V$.

Problem 2. The *ordered pair* of sets x and y is defined as

$$(x, y) \;=\; \{\{x\}, \{x, y\}\}.$$

(1) (1 point) Prove that (x, y) exists for all sets x and y.
(2) (2 points) Prove that

$$\forall x \; \forall x' \; \forall y \; \forall y' \; [(x, y) = (x', y') \;\longleftrightarrow\; (x = x \wedge y = y')].$$

(3) (1 point) Construct an injective class function $P : V \times V \longrightarrow V$ with $F(x, y) \neq (x, y)$ for all sets x and y.

Problem 3. Given sets x and y, we define

$$x_\Delta y \;=\; x \setminus y \;\cup\; y \setminus x.$$

(1) (3 points) Given a nonempty set x, show the structure

$$\mathcal{B}_x \;=\; \langle \mathcal{P}(x), \emptyset, x, \Delta, \cap \rangle$$

is a ring.
(2) (2 point) Let $\mathbf{F}_2$ denote the field with two elements. Given a nonempty set x, show that the ring $\mathcal{B}_x$ is an $\mathbf{F}_2$-algebra.
(3) (1 point) Determine the class of all nonempty sets x with the property that $\mathcal{B}_x$ is a field.

Problem 4. A class A is *transitive*, if every element of A is a subset of A.

(1) (1 point) Show that the classes $\emptyset$ and V are transitive.
(2) (2 bonus points) Show that the *Axiom of Foundation* implies that a transitive class is either empty or contains the empty set.

(3) (2 point) Given a class A consisting of transitive sets, show that the classes $\bigcup A$ and $\bigcap A$ are transitive.

(4) (1 points) Given a transitive class A, show that the class $\bigcup A$ is transitive.

(5) (2 bonus points) Show that the *Axiom of Foundation* implies that the class $\bigcap A$ is transitive for every transitive class A.

(6) (2 points) Show that the class of all transitive sets is not transitive.

Please hand in your solutions on Wednesday, October 17 before the lecture (Briefkästen 6 & 7).