Models of Set Theory II - Winter 2017/2018

Prof. Dr. Peter Koepke, Ana Njegomir Problem sheet 4

Problem 1 (6 points)**.** Let κ be a cardinal such that $\kappa = \kappa^\omega$. We say that $\mathrm{FA}_\kappa(\aleph_1\text{-closed})$ holds if and only if $\mathrm{FA}_\kappa(\mathbb{Q})$ holds for every forcing notion $\mathbb{Q}$ that is $\aleph_1$-closed. Prove that $\mathrm{FA}_\kappa(\aleph_1\text{-closed})$ is equivalent to $\mathrm{FA}_\kappa(\aleph_1\text{-closed})$ restricted to partial orders of cardinality $\leq \kappa$.

Problem 2 (6 points)**.** Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing notions. Prove that the following statements are equivalent:

(1) $\mathbb{P} \times \mathbb{Q}$ is ccc.

(2) $\mathbb{P}$ is ccc and $\mathbb{1}_\mathbb{P} \Vdash^M_\mathbb{P}$ "$\check{\mathbb{Q}}$ is ccc".

(3) $\mathbb{Q}$ is ccc and $\mathbb{1}_\mathbb{Q} \Vdash^M_\mathbb{Q}$ "$\check{\mathbb{P}}$ is ccc".

Hint: For "(1) $\to$ (2)" assume $p \Vdash^M_\mathbb{P}$ "$\dot{f} : \check{\omega}_1 \to \check{\mathbb{Q}}$ enumerates an antichain" and choose a suitable antichain in $\mathbb{P}$ below p. For the converse, consider the $\mathbb{P}$-name $\sigma = \{\langle \check{\xi}, p_\xi \rangle \mid \xi < \omega_1\}$ and show that whenever G is M-generic for $\mathbb{P}$, σ^G is countable.

Problem 3 (4 points)**.** Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing notions. Prove from MA_{ω_1} that $\mathbb{P} \times \mathbb{Q}$ is ccc if and only if $\mathbb{P}$ and $\mathbb{Q}$ are ccc.

Definition. We say that the Diamond principle $\diamondsuit$ holds if there exists a sequence of sets $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ with $S_\alpha \subset \alpha$, such that for every $X \subset \omega_1$, the set $\{\alpha < \omega_1 \mid X \cap \alpha = S_\alpha\}$ is a stationary subset of ω_1.

Problem 4 (6 points)**.** Suppose that CH holds in a ground model M. Let $\mathbb{P} = \{\langle S_\alpha \mid \alpha < \delta \rangle \mid S_\alpha \subseteq \alpha, \delta < \omega_1\} \in M$ be a forcing notion ordered by reverse inclusion and let G be a $\mathbb{P}$-generic filter over M. Prove that the Diamond principle $\diamondsuit$ holds in $M[G]$.

Please hand in your solutions on Monday, November 6 before the lecture.