Models of Set Theory II - Winter 2017/2018

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Definition. A *tree* is a strict partial order $(\mathbb{T}, <_\mathbb{T})$ such that the following hold.

(1) $(\mathbb{T}, <_\mathbb{T})$ has a unique minimal element $\mathrm{root}(\mathbb{T})$.

(2) The set $\mathrm{pred}_\mathbb{T}(t) = \{s \in \mathbb{T} \mid s <_\mathbb{T} t\}$, is well-ordered by $<_\mathbb{T}$ for every $t \in \mathbb{T}$.

Definition. Let $(\mathbb{T}, <_\mathbb{T})$ be a tree.

(1) Given $t \in \mathbb{T}$, we define the length of t, $\mathrm{lh}_\mathbb{T}(t)$, to be the order-type of $pred_\mathbb{T}(t)$.

(2) Given $\alpha \in \mathrm{Ord}$, we define the α-th level of $\mathbb{T}$ by $\mathbb{T}(\alpha) = \{t \in \mathbb{T} \mid \mathrm{lh}_\mathbb{T}(t) = \alpha\}$.

(3) Define the height of $(\mathbb{T}, <_\mathbb{T})$ to be $\mathrm{ht}(\mathbb{T}) = \min\{\alpha \in \mathrm{Ord} \mid \mathbb{T}(\alpha) = \emptyset\}$.

(4) A branch through $\mathbb{T}$ is a maximal linearly ordered subset of $\mathbb{T}$. The length of a branch b, $\mathrm{lh}_\mathbb{T}(b)$, is the order-type of it.

(5) A cofinal branch through $\mathbb{T}$ is a branch b through $\mathbb{T}$ with $\mathrm{lh}_\mathbb{T}(b) = \mathrm{ht}(\mathbb{T})$.

Problem 1 (6 points)**.** Let κ be an infinite cardinal. A κ-*Aronszajn tree* is a tree $(\mathbb{T}, <_\mathbb{T})$ with $\mathrm{ht}(\mathbb{T}) = \kappa$, $|\mathbb{T}(\alpha)| < \kappa$ for all $\alpha < \kappa$ and without cofinal branches. We say that κ has the tree property if there are no κ-Aronszajn trees. Prove that the cardinal $\omega = \aleph_0$ has the tree property.

Definition. Let κ be an uncountable regular cardinal and $\mathbb{T}$ be a tree of height κ.

(1) A function $r : \mathbb{T} \to \mathbb{T}$ is called *regressive* if $r(t) <_\mathbb{T} t$ holds for every $t \in \mathbb{T} \setminus \{\mathrm{root}(\mathbb{T})\}$.

(2) The tree $\mathbb{T}$ is called *special* if there is a regressive map $r : \mathbb{T} \to \mathbb{T}$ with the property that $r^{-1}[\{t\}]$ is the union of less than κ-many antichains in $\mathbb{T}$, for every $t \in \mathbb{T}$. In other words, for every $t \in \mathbb{T}$ there is some $\lambda < \kappa$ and a function $c_t : r^{-1}[\{t\}] \to \lambda$ such that $c_t(s_0) \neq c_t(s_1)$ for all $s_0, s_1 \in \mathbb{T}$ with $r(s_0) = r(s_1) = t$ and $s_0 \leq_\mathbb{T} s_1$.

Problem 2 (4 points)**.** Let κ be an uncountable regular cardinal and let $\mathbb{T}$ be a tree of height κ. Show that if $\mathbb{T}$ is special, then $\mathbb{T}$ does not have cofinal branches.

Problem 3 (6 points)**.** Let M be a transitive and countable model of ZFC. Two forcing notions $\mathbb{P} \in M$ and $\mathbb{Q} \in M$ are said to be *forcing equivalent* over M, if every $\mathbb{P}$-generic extension of M is also a $\mathbb{Q}$-generic extension of M, and vice versa every $\mathbb{Q}$-generic extension of M is also a $\mathbb{P}$-generic extension of M. Let $\mathbb{P}$ be the forcing notion whose conditions are intervals $[p, q]$ of the real line $\mathbb{R}$ with rational endpoints $p < q$, partially ordered by inclusion.

(a) Let G be M-generic for $\mathbb{P}$. Prove that G can be reconstructed from a single real.

(b) Show that $\mathbb{P}$ is forcing equivalent to Cohen forcing $\mathbb{C}$.

Please hand in your solutions on Monday, October 16 before the lecture.