Models of Set Theory II - Winter 2017/2018

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Definition. A tree $\mathbb{T}$ is a *Suslin tree* if the following holds.

(1) The height of $\mathbb{T}$ is ω_1.
(2) Every branch in $\mathbb{T}$ is at most countable.
(3) Every antichain in $\mathbb{T}$ is at most countable.

Definition. Let $\alpha \in \mathrm{Ord}$ such that $\alpha \leq \omega_1$. A normal α-tree is a tree $\mathbb{T}$ with the following properties:

(1) $\mathrm{ht}(\mathbb{T}) = \alpha$,
(2) each level of $\mathbb{T}$ is at most countable,
(3) if x is not maximal in $\mathbb{T}$, then there are infinitely many $y > x$ at the the next level,
(4) for each $x \in \mathbb{T}$ there is some $y > x$ at each higher level less than α,
(5) if $\beta < \alpha$ is a limit ordinal and x, y are both at level β and if $\mathrm{pred}_{\mathbb{T}}(x) = \mathrm{pred}_{\mathbb{T}}(y)$, then $x = y$.

Definition. Define $\mathbb{P}_{\mathbb{T}}$ to be the collection of all countable normal trees. In other words, $\mathbb{T} \in \mathbb{P}_{\mathbb{T}}$ if and only if for some $\alpha < \omega_1$,

(1) each $t \in \mathbb{T}$ is a function $t : \beta \to \omega$ for some $\beta < \alpha$;
(2) if $t \in \mathbb{T}$ and s is an initial segment of t, then $s \in \mathbb{T}$;
(3) if $\beta + 1 < \alpha$ and $t : \beta \to \omega$ is in $\mathbb{T}$, then $t^\frown n \in \mathbb{T}$ for all $n \in \omega$.
(4) if $\beta < \alpha$ and $t : \beta \to \omega$ is in $\mathbb{T}$, then for every γ such that $\beta \leq \gamma < \alpha$ there exists an $s : \gamma \to \omega$ in $\mathbb{T}$ such that $t \subset s$.
(5) $\mathbb{T} \cap \omega^\beta$ is at most countable for all $\beta < \alpha$

$\mathbb{T}_1$ is stronger than $\mathbb{T}_2$ if $\mathbb{T}_1$ is an extension of $\mathbb{T}_2$. In other words $\mathbb{T}_1 < \mathbb{T}_2$ if and only if there exists $\alpha < \mathrm{ht}(\mathbb{T}_1)$ such that $\mathbb{T}_2 = \{t \restriction \alpha \mid t \in \mathbb{T}_1\}$.

Problem 1 (2 points)**.** A set $S \subset \mathbb{T}$ is *bounded* in a tree $\mathbb{T}$ if there is some $\alpha < \mathrm{ht}(\mathbb{T})$ such that all elements of S are at levels $\leq \alpha$. If A is a maximal antichain in a normal tree $\mathbb{T}$ and if A is bounded in $\mathbb{T}$, then A is maximal in every extension of $\mathbb{T}$.

Problem 2 (2 points)**.** Let α be a countable limit ordinal. Suppose that $\mathbb{T} \in \mathbb{P}_{\mathbb{T}}$ is a normal α-tree and let A be a maximal antichain in $\mathbb{T}$. Prove that there is an extension $T^* \in \mathbb{P}_{\mathbb{T}}$ of $\mathbb{T}$ of height $\alpha + 1$ such that A is a maximal antichain in T^*.

Problem 3 (6 points)**.** Let G be a $\mathbb{P}_{\mathbb{T}}$-generic over M. Prove that in $M[G]$ there is a Suslin tree.

Hint: First show that $S = \bigcup G$ is a normal ω_1-tree. Then show that S has no uncountable antichain: Pick a name $\dot{A}$ for some maximal antichain A in S. Let $\mathbb{T} \in G$ forces that $\dot{A}$ is a maximal antichain in S. Complete the proof by showing that the set all $\mathbb{T}' \leq \mathbb{T}$ such that there is a bounded maximal antichain $A' \in \mathbb{T}'$ with $\mathbb{T}' \Vdash \dot{A}' \subset \dot{A}$, where $\dot{A}'$ is a name for A', is dense below $\mathbb{T}$. Problems 1 and 2 will be usefull for this.

Problem 4 (6 points)**.** Prove that if there exists a Suslin tree then there exists a normal Suslin tree.

Problem 5 (4 points)**.** Let $M \prec N$ be models of ZFC which have the same cardinals. Suppose that $\mathbb{P} \in M$ is a partial order. Prove that antichains in $\mathbb{P}$, maximal antichains in $\mathbb{P}$, dense subsets of $\mathbb{P}$, $\mathbb{P}$ having ccc and being a $\mathcal{D}$-generic filter on $\mathbb{P}$, where $\mathcal{D}$ is some collection of dense sets, is absolute between those two models.

Please hand in your solutions on Monday, November 13 before the lecture.