Models of Set Theory II - Winter 2017/2018

Prof. Dr. Peter Koepke, Ana Njegomir Problem sheet 3

Problem 1 (6 points)**.** Let M be a transitive and countable model of ZFC. If $\mathbb{P}$ is a forcing notion and G is M-generic for $\mathbb{P}$, we say that $f \in ({}^{\omega}\omega)^{M[G]}$ is a *dominating real*, if for every $g \in ({}^{\omega}\omega)^M$ there is $n_0 \in \omega$ such that for all $n \geq n_0$, $g(n) < f(n)$. Prove that if $M[G]$ contains a dominating real then it contains a splitting real.

Hint: If $f \in M[G]$ is dominating, one may assume that f is strictly increasing. Let $f^{n+1}(0) = f(f^n(0))$ and $f^0(0) = 0$ and construct a splitting real by taking the union of certain intervals of the form $[f^k(0), f^l(0))$ in a suitable way, where for $n, m \in \omega$, $[n, m) = \{k \in \omega \mid n \leq k < m)\}$.

Problem 2 (6 points)**.** Let $\mathbb{C}$ denote Cohen forcing. Prove that $\mathbb{C}$ does not add dominating reals.

Hint Consider an enumeration $\langle p_n \mid n \in \omega \rangle$ of the conditions in $\mathbb{C}$ and $g(n) = \min\{k \in \omega \mid \exists p \leq_{\mathbb{C}} p_n (p \Vdash_{\mathbb{C}} \dot{f}(\check{n}) = \check{k})\}$ for a $\mathbb{C}$-name $\dot{f}$ for a real.

Problem 3 (8 points)**.** *Hechler forcing* $\mathbb{P}$ is the forcing notion whose conditions are of the form $p = \langle s_p, E_p \rangle$ such that $s_p : n \to \omega$ for some $n \in \omega$ and $E_p \subseteq {}^{\omega}\omega$ is a finite set of functions from ω to ω. The ordering is given by

$$p \leq_{\mathbb{P}} q \iff s_p \supseteq s_q \wedge E_p \supseteq E_q \wedge \forall f \in E_q \, \forall n \in \operatorname{dom}(s_p) \setminus \operatorname{dom}(s_q)(f(n) < s_p(n)).$$

(a) Show that $\mathbb{P}$ satisfies the c.c.c.

(b) Assume that MA holds. Let $F \subset {}^{\omega}\omega$ and $\operatorname{card}(F) < 2^{\aleph_0}$. Prove that there is $f \in \omega^{\omega}$ such that for every $g \in F$ there is $n_0 \in \omega$ such that for all $n \geq n_0$, $g(n) < f(n)$.

Please hand in your solutions on Monday, October 30 before the lecture.