Models of Set Theory I

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Abstract

Transitive models of set theory, the relative consistency of the axiom of choice using the hereditarily ordinal definable sets, forcing conditions and generic filters, generic extensions, ZFC holds in generic extensions, the relative consistency of the continuum hypothesis and of the negation of the continuum hypothesis, possible behaviours of the function 2^κ, the relative consistency of the negation of the axiom of choice.

1 Introduction

Sets are axiomatized by the Zermelo-Fraenkel axiom system ZF. Following Jech [?] these axioms can be formulated in the first-order language with one binary relation symbol $\in$ as

- *Extensionality*: $\forall z(z\in x\leftrightarrow z\in y)\rightarrow x=y$
- *Pairing*: $\exists z\forall u(u\in z\leftrightarrow u=x\vee u=y)$
- *Union*: $\exists z\forall u(u\in z\leftrightarrow\exists y(u\in y\wedge y\in x))$
- *Power*: $\exists z\forall u(u\in z\leftrightarrow\forall v(v\in u\rightarrow u\in x))$
- *Infinity*: $\exists z(\exists x(x\in z\wedge\forall y\neg y\in x)\wedge\forall u(u\in z\rightarrow\exists v(v\in z\wedge\forall w(w\in v\leftrightarrow w\in u\vee w= u))))$
- *Separation*: for every $\in$-formula $\varphi(u,p)$ postulate $\exists z\forall u(u\in z\leftrightarrow u\in x\wedge\varphi(u,p))$
- *Replacement*: for every $\in$-formula $\varphi(u,v,p)$ postulate

$$\forall u,v,v'(\varphi(u,v,p)\wedge\varphi(u,v',p)\rightarrow v=v')\rightarrow\exists y\forall v(v\in y\leftrightarrow\exists u(u\in x\wedge\varphi(u,v,p)))$$

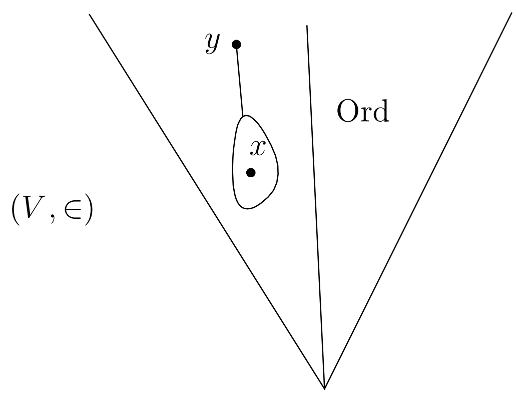
- *Foundation*: $\exists u\, u\in x\rightarrow\exists u(u\in x\wedge\forall v(v\in u\rightarrow\neg v\in x))$

The axioms capture the basic intuitions of Cantorean set theory. They are strong enough to formalise all other mathematical fields. Usually the *Axiom of Choice* is also assumed

- *Choice* or AC: $\forall u,u'((u\in x\rightarrow\exists v\, v\in u)\wedge(u\in x\wedge u'\in x\wedge u\neq u'\rightarrow\neg\exists v(v\in u\wedge v\in u')))\rightarrow\exists y\forall u(u\in x\rightarrow\exists v(v\in u\wedge v\in y\wedge\forall v'(v'\in u\wedge v'\in y\rightarrow v'=v))))$.

ZFC is the system consisting of ZF and AC. ZF^- consists of all ZF-axioms except the powerset axiom.

We use the intuition of a standard model of set theory $(V, \in)$, the *universe* of all (mathematical) sets. This is usually pictured like an upward open triangle with the understanding that if $x \in y$ then x lies below y; x is in the *extension* of y. The ordinals are pictured by a central line, extending to infinity.



Although this picture gives some useful intuition, we can only know about sets by deduction from the ZF-axioms. On the other hand the axioms are incomplete in that they do not decide important properties of infinitary combinatorics. The most important examples are

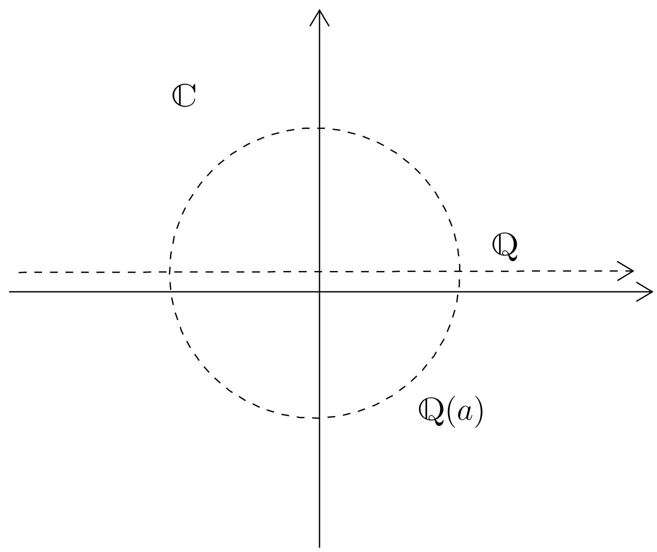
- the system ZF does not decide the axiom of choice AC: if ZF is a consistent theory, then so are $\mathrm{ZF} + \mathrm{AC}$ and $\mathrm{ZF} + \neg\mathrm{AC}$

- the system ZFC does not decide the continuum hypothesis: if ZFC is a consistent theory, then so are $\mathrm{ZFC} + \mathrm{CH}$ and $\mathrm{ZFC} + \neg\mathrm{CH}$

Here a theory is *consistent*, if it does not imply a contradiction like $x \neq x$.

We appeal to the following central fact from mathematical logic: a theory T is consistent iff it possesses a model. This allows to show consistency results by constructing models of ZF and of ZFC.

We motivate the construction methods by analogy with the construction of fields in algebra. The complex numbers $(\mathbb{C}, +, \cdot, 0, 1)$ form a standard field for many purposes.



$\mathbb{C}$ is an algebraically closed field. It contains (isomorphic copies of) many interesting fields, like the rationals $\mathbb{Q}$, or extensions of $\mathbb{Q}$ of finite degree (algebraic number fields). These subfields witness consistency results for the theory of fields:

- the field axioms do not decide the existence of $\sqrt{2}$: $\mathbb{Q}$ is a model of $\neg\exists x\, x\cdot x = 1+1$, whereas $\mathbb{Q}(\sqrt{2})$ is a model of $\exists x\, x\cdot x = 1+1$

- by successively adjoining square roots one can form a field which satisfies $\forall y \exists x\, x\cdot x = y$ but which does not contain $\sqrt[3]{2}$. This is used to show that the doubling of the cube cannot be performed by ruler and compass

Let us mention a few properties of field constructions which will have analogues in constructions of models of set theory

- the fields are (or can be) embedded into the standard field $\mathbb{C}$.

- the extension fields $k(a)$ can be described within the ground field k: a is either algebraic or transcendental over k; in the algebraic case one can treat a as a variable x which is a zero of some polynomial in $k[x]$: $p(x) = 0$; in the transcendental case a corresponds to a variable x such that $p(x) \neq 0$ for all $p \in k[x]$; calculations in $k(a)$ can be reduced to calculations in k.

- the ground field $\mathbb{Q}$ is countable. One can construct a transcendental real

$$a = 0, a_0 a_1 a_2 a_3 \ldots \in \mathbb{R}$$

by successively choosing decimals a_i so that $0,a_0a_1...a_m$ "forces" $p_n(a)\neq 0$, i.e.,

$$\forall b\,(b=0,a_0a_1a_2a_3...a_mb_{m+1}b_{m+1}...\to p_n(b)\neq 0).$$

Here $(p_n)_{n<\omega}$ is some enumeration of $k[x]$. In view of the *forcing method* in set theory we can write this as

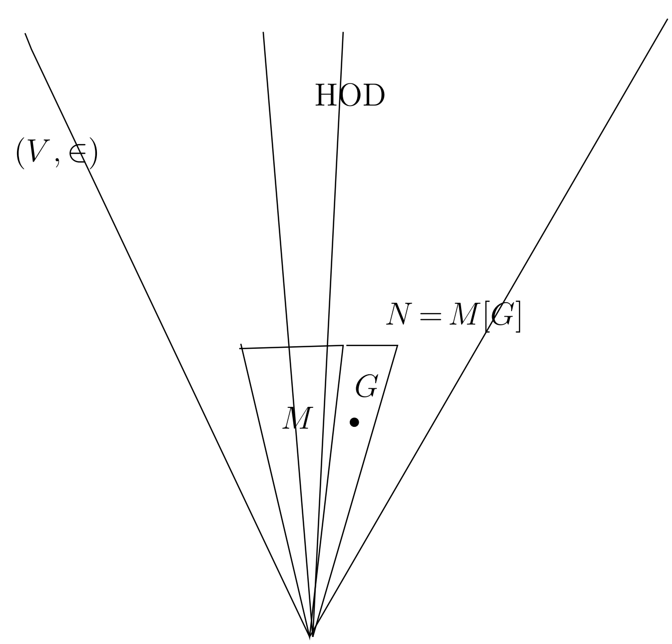
$$0,a_0a_1a_2a_3...a_m \Vdash p_n(\dot{x})\neq 0$$

where $\dot{x}$ is a symbol or *name* for the transcendental or *generic* real to be constructed.

For models of set theory this translates to

- consider transitive submodels $(M,\in)$ of the standard universe $(V,\in)$.
- construct minimal submodels similar to the prime field $\mathbb{Q}$.
- construct *generic extensions* $N\supseteq M$ by adjoining *generic* sets G, corresponding to the transcendental numbers above: $N=M[G]$.
- G is describable in the countable ground model M by infinitely many formulas, it will be constructed by a countable recursion along countably many requirements which can be expressed inside M.

We shall consider the models HOD (*Hereditarily Ordinal Definable sets*), generic extensions $M[G]$, and *symmetric submodels* N of $M[G]$. This leads to a spectrum $\mathrm{HOD}, L, M, M[G], N...$ of models of set theory like



These models satisfy different extensions of the ZF-axioms: e.g., $\mathrm{HOD} \vDash \mathrm{AC}$, $M[G]$ may satisfy CH or $\neg$CH, and symmetric submodels may satisfy $\neg$AC. This leads to the desired (relative) consistency results.

2 Transitive Models of Set Theory

Let W be a transitive class. We consider situations when W together with the $\in$-relation restricted to W is a model of axioms of set theory. So we are interested in the "model" $(W, \in)$ or $(W, \in\upharpoonright W)$ where $\in\upharpoonright W = \{(u, v) | u \in v \in W\}$. Considering W as a universe for set theory means that the quantifiers $\forall$ and $\exists$ in $\in$-formulas φ range over W instead over the full universe V. For simplicity we assume that $\in$-formulas are only formed by variables $v_0, v_1, \ldots$, the relations $=$ and $\in$, and logical signs $\neg$, $\vee$, $\exists$.

Definition 1. *Let W be a term and φ be an $\in$-formula which do not have common variables. The* relativisation *φ^W of φ to W is defined recursively along the structure of φ:*

- $(v_i \in v_j)^W \equiv (v_i \in v_j)$
- $(v_i = v_j)^W \equiv (v_i = v_j)$
- $(\neg\varphi)^W \equiv \neg(\varphi^W)$
- $(\varphi \vee \psi)^W \equiv ((\varphi^W) \vee (\psi^W))$
- $(\exists v_i\, \varphi)^W \equiv \exists v_i \in W\ (\varphi^W)$

If Φ is a collection of $\in$-formulas set $\Phi^W = \{\varphi^W | \varphi \in \Phi\}$. Instead of φ^W or Φ^W we also say "φ holds in W", "Φ holds in W", "W is a model of φ", etc.; we also write $W \vDash \varphi$ and $W \vDash \Phi$.

φ^W and Φ^W are obtained from φ and Φ by bounding all quantifiers by the class W.

We prove criteria for set theoretic axioms to hold in W.

Theorem 2. *Assume* ZF*. Let W be a transitive class, $W \neq \emptyset$. Then*

a) $(\mathit{Extensionality})^W$.

b) $(\mathit{Pairing})^W \leftrightarrow \forall x \in W \forall y \in W\ \{x, y\} \in W$.

c) $(\mathit{Union})^W \leftrightarrow \forall x \in W \bigcup x \in W$.

d) $(\mathit{Power})^W \leftrightarrow \forall x \in W \mathcal{P}(x) \cap W \in W$.

e) $(\mathit{Infinity})^W \leftrightarrow \exists z \in W\, (\emptyset \in z \wedge \forall u \in z\;\, u+1 \in z)$.

f) *Let* ψ *be the instance of the Separation schema for the* $\in$*-formula* $\varphi(x, \vec{w})$*. Then*

$$\psi^W \leftrightarrow \forall \vec{w} \in W \forall a \in W\; \{x \in a | \varphi^W(x, \vec{w})\} \in W.$$

g) *Let* ψ *be the instance of the Replacement schema for the* $\in$*-formula* $\varphi(x,\ y,\ \vec{w})$*. Then* ψ^W *is equivalent to*

$\forall \vec{w} \in W(\forall x,\ y,\ y' \in W(\varphi^W(x,\ y,\ \vec{w}) \wedge \varphi^W(x,\ y',\ \vec{w}) \rightarrow y = y') \rightarrow \forall a \in W\{y | \exists x \in a \varphi^W(x, y, \vec{w})\} \cap W \in W)$.

h) $(\mathit{Foundation})^W$.

i) $(\mathit{Choice})^W \leftrightarrow \forall x \in W(\emptyset \notin x \wedge \forall u, u' \in x(u \neq u' \rightarrow u \cap u' = \emptyset) \rightarrow \exists y \in W \forall u \in x \exists v\; \{v\} = u \cap y)$.

Proof. *Bounded* quantications are not affected by relativisations to *transitive* classes:

(1) Let $x \in W$. Then $\forall y(y \in x \rightarrow \varphi) \leftrightarrow \forall y \in W(y \in x \rightarrow \varphi)$ and $\exists y(y \in x \wedge \varphi) \leftrightarrow \exists y \in W(y \in x \wedge \varphi)$.

Proof. Assume that $\forall y \in W(y \in x \rightarrow \varphi)$. To show $\forall y(y \in x \rightarrow \varphi)$ consider some $y \in x$. By the transitivity of W, $y \in W$. By assumption, φ holds. *qed*(1)

a)

$$\begin{aligned}
(\text{Extensionality})^W &\leftrightarrow (\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y))^W \\
&\leftrightarrow \forall x \in W \forall y \in W[\forall z \in W(z \in x \leftrightarrow z \in y) \rightarrow x = y] \\
&\leftrightarrow \forall x \in W \forall y \in W[[\forall z \in W(z \in x \rightarrow z \in y) \wedge \forall z \in W(z \in y \rightarrow z \in x)] \rightarrow \\
&\quad\; x = y] \\
&\leftrightarrow \forall x \in W \forall y \in W[[\forall z(z \in x \rightarrow z \in y) \wedge \forall z(z \in y \rightarrow z \in x)] \rightarrow x = y], \text{ by} \\
&\quad\; (1).
\end{aligned}$$

The righthand side is a consequence of Extensionality in V.

b)

$$\begin{aligned}
(\text{Pairing})^W &\leftrightarrow (\forall x \forall y \exists z \forall u(u \in z \leftrightarrow u = x \vee u = y))^W \\
&\leftrightarrow \forall x \in W \forall y \in W \exists z \in W \forall u \in W(u \in z \leftrightarrow u = x \vee u = y) \\
&\leftrightarrow \forall x \in W \forall y \in W \exists z \in W \forall u(u \in z \leftrightarrow u = x \vee u = y), \text{ by (1)} \\
&\leftrightarrow \forall x \in W \forall y \in W \exists z \in W\, z = \{x, y\} \\
&\leftrightarrow \forall x \in W \forall y \in W\, \{x, y\} \in W.
\end{aligned}$$

c)

$$\begin{aligned}
(\text{Union})^W &\leftrightarrow (\forall x \exists z \forall u (u \in z \leftrightarrow \exists y (u \in y \wedge y \in x)))^W \\
&\leftrightarrow \forall x \in W \exists z \in W \forall u \in W (u \in z \leftrightarrow \exists y \in W (u \in y \wedge y \in x)) \\
&\leftrightarrow \forall x \in W \exists z \in W \forall u \in W (u \in z \leftrightarrow \exists y (u \in y \wedge y \in x)), \text{ by (1)} \\
&\leftrightarrow \forall x \in W \exists z \in W \forall u (u \in z \leftrightarrow \exists y (u \in y \wedge y \in x)), \text{ by (1)} \\
&\leftrightarrow \forall x \in W \exists z \in W\, z = \bigcup x \\
&\leftrightarrow \forall x \in W \bigcup x \in W
\end{aligned}$$

d)

$$\begin{aligned}
(\text{Power})^W &\leftrightarrow (\forall x \exists z \forall u (u \in z \leftrightarrow \forall v (v \in u \to u \in x)))^W \\
&\leftrightarrow \forall x \in W \exists z \in W \forall u \in W (u \in z \leftrightarrow \forall v \in W (v \in u \to u \in x)) \\
&\leftrightarrow \forall x \in W \exists z \in W \forall u \in W (u \in z \leftrightarrow \forall v (v \in u \to u \in x)), \text{ by (1)} \\
&\leftrightarrow \forall x \in W \exists z \in W \forall u \in W (u \in z \leftrightarrow u \subseteq x) \\
&\leftrightarrow \forall x \in W \exists z \in W \forall u (u \in z \leftrightarrow u \in W \wedge u \subseteq x) \\
&\leftrightarrow \forall x \in W \exists z \in W\, z = \mathcal{P}(x) \cap W \\
&\leftrightarrow \forall x \in W\, \mathcal{P}(x) \cap W \in W
\end{aligned}$$

e)

$$\begin{aligned}
(\text{Infinity})^W &\leftrightarrow (\exists z (\exists x (x \in z \wedge \forall y \neg y \in x) \wedge \forall u (u \in z \to \exists v (v \in z \wedge \forall w (w \in v \leftrightarrow w \in u \vee w = \\
&\quad u)))))^W \\
&\leftrightarrow \exists z \in W (\exists x \in W (x \in z \wedge \forall y \in W \neg y \in x) \wedge \forall u \in W (u \in z \to \exists v \in W (v \in z \wedge \\
&\quad \forall w \in W (w \in v \leftrightarrow w \in u \vee w = u)))) \\
&\leftrightarrow \exists z \in W (\exists x (x \in z \wedge \forall y \neg y \in x) \wedge \forall u (u \in z \to \exists v (v \in z \wedge \forall w (w \in v \leftrightarrow w \in u \vee \\
&\quad w = u)))), \text{ by (1)} \\
&\leftrightarrow \exists z \in W (\emptyset \in z \wedge \forall u (u \in z \to u + 1 \in z)).
\end{aligned}$$

f) Separation:

$$\begin{aligned}
(\forall \vec{w} \forall a \exists y \forall x (\, x \in y \leftrightarrow x \in a \wedge \varphi(x, \vec{w})))^W &\leftrightarrow \forall \vec{w} \in W \forall a \in W \exists y \in W \forall x \in W (\, x \in y \leftrightarrow x \in a \wedge \\
&\quad \varphi^W(x, \vec{w})) \\
&\leftrightarrow \forall \vec{w} \in W \forall a \in W \exists y \in W \forall x (\, x \in y \leftrightarrow x \in a \wedge \\
&\quad \varphi^W(x, \vec{w})), \text{ by (1)} \\
&\leftrightarrow \forall \vec{w} \in W \forall a \in W \exists y \in W\, y = \{x \in a | \varphi^W(x, \vec{w})\} \\
&\leftrightarrow \forall \vec{w} \in W \forall a \in W\, \{x \in a | \varphi^W(x, \vec{w})\} \in W
\end{aligned}$$

g) Replacement:

$$\begin{aligned}\psi^W &= (\forall\vec{w}(\forall x,y,y'(\varphi(x,y,\vec{w})\wedge\varphi(x,y',\vec{w})\rightarrow y=y')\rightarrow\forall a\exists z\forall y(y\in z\leftrightarrow\exists x(x\in a\wedge\varphi(x,\\&\quad y,\vec{w})))))^W\\
&\leftrightarrow \forall\vec{w}\in W(\forall x,y,y'\in W(\varphi^W(x,y,\vec{w})\wedge\varphi^W(x,y',\vec{w})\rightarrow y=y')\rightarrow\forall a\in W\exists z\in W\forall y\in\\&\quad W(y\in z\leftrightarrow\exists x\in W(x\in a\wedge\varphi^W(x,y,\vec{w}))))\\
&\leftrightarrow \forall\vec{w}\in W(\forall x,y,y'\in W(\varphi^W(x,y,\vec{w})\wedge\varphi^W(x,y',\vec{w})\rightarrow y=y')\rightarrow\forall a\in W\exists z\in W\forall y(y\in\\&\quad z\leftrightarrow(\exists x(x\in a\wedge\varphi^W(x,y,\vec{w}))\wedge y\in W))\\
&\leftrightarrow \forall\vec{w}\in W(\forall x,y,y'\in W(\varphi^W(x,y,\vec{w})\wedge\varphi^W(x,y',\vec{w})\rightarrow y=y')\rightarrow\forall a\in W\exists z\in W z=\\&\quad \{y|\exists x\in a\,\varphi^W(x,y,\vec{w}))\}\cap W)\\
&\leftrightarrow \forall\vec{w}\in W(\forall x,y,y'\in W(\varphi^W(x,y,\vec{w})\wedge\varphi^W(x,y',\vec{w})\rightarrow y=y')\rightarrow\forall a\in W\ \{y|\exists x\in\\&\quad a\,\varphi^W(x,y,\vec{w}))\}\cap W\in W).\end{aligned}$$

h)

$$\begin{aligned}(\text{Foundation})^W &\leftrightarrow (\forall x(\exists u\, u\in x\rightarrow\exists u(u\in x\wedge\forall v(v\in u\rightarrow\neg v\in x))))^W\\
&\leftrightarrow \forall x\in W(\exists u\in W u\in x\rightarrow\exists u\in W(u\in x\wedge\forall v\in W(v\in u\rightarrow\neg v\in x)))\\
&\leftrightarrow \forall x\in W(\exists u\, u\in x\rightarrow\exists u(u\in x\wedge\forall v(v\in u\rightarrow\neg v\in x))),\text{ by (1).}\\
&\leftarrow \forall x(\exists u\, u\in x\rightarrow\exists u(u\in x\wedge\forall v(v\in u\rightarrow\neg v\in x)))\\
&\leftrightarrow \text{Foundation in } V.\end{aligned}$$

i) Choice:

$$\begin{aligned}\mathrm{AC}^W &\leftrightarrow (\forall x(\forall u,u'((u\in x\rightarrow\exists v\, v\in u)\wedge(u\in x\wedge u'\in x\wedge u\neq u'\rightarrow\neg\exists v(v\in u\wedge v\in\\&\quad u')))\rightarrow\exists y\forall u(u\in x\rightarrow\exists v(v\in u\wedge v\in y\wedge\forall v'(v'\in u\wedge v'\in y\rightarrow v'=v))))))^W\\
&\leftrightarrow \forall x\in W(\forall u,u'\in W((u\in x\rightarrow\exists v\in W v\in u)\wedge(u\in x\wedge u'\in x\wedge u\neq u'\rightarrow\neg\exists v\in\\&\quad W(v\in u\wedge v\in u')))\rightarrow\exists y\in W\forall u\in W(u\in x\rightarrow\exists v\in W(v\in u\wedge v\in y\wedge\forall v'\in W(v'\in\\&\quad u\wedge v'\in y\rightarrow v'=v)))))\\
&\leftrightarrow \forall x\in W(\forall u,u'((u\in x\rightarrow\exists v\, v\in u)\wedge(u\in x\wedge u'\in x\wedge u\neq u'\rightarrow\neg\exists v(v\in u\wedge v\in\\&\quad u')))\rightarrow\exists y\in W\forall u(u\in x\rightarrow\exists v(v\in u\wedge v\in y\wedge\forall v'(v'\in u\wedge v'\in y\rightarrow v'=v))))),\text{ by}\\&\quad\text{several applications of (1),}\\
&\leftrightarrow \forall x\in W(\emptyset\notin x\wedge\forall u,u'\in x(u\neq u'\rightarrow u\cap u'=\emptyset)\rightarrow\exists y\in W\forall u\in x\exists v\,\{v\}=u\cap y)\end{aligned}$$

□

The theorem yields models of fragments of ZFC in the von Neumann hierarchy $(V_\alpha)_{\alpha\in\mathrm{Ord}}$.

Theorem 3. *Assume* ZF. *Then*

a) $V_\alpha\vDash$ *Extensionality, Union, Separation, and Foundation;*

b) *if α is a limit ordinal then $V_\alpha\vDash$ Pairing and Powerset;*

c) *if* $\alpha > \omega$ *then* $V_\alpha \vDash$ *Infinity;*

d) *if* AC *holds then* $V_\alpha \vDash \mathrm{AC}$*;*

e) *if* AC *holds,* α *is a regular limit ordinal and* $\forall\lambda < \alpha\, 2^\lambda < \alpha$, *then* $V_\alpha \vDash$ *Replacement;*

f) $V_\omega \vDash$ *all axioms of* ZFC *except Infinity;*

g) *if* AC *holds and* α *is strongly inaccessible, i.e.* α *is a regular limit ordinal* $>\omega$ *and* $\forall\lambda < \alpha\, 2^\lambda < \alpha$ *then* $V_\alpha \vDash \mathrm{ZFC}$.

Proof. *e*) First prove by induction on $\xi \in [\omega, \alpha)$ that $\forall a \in V_\xi\ \mathrm{card}(a) < \alpha$. For the replacement criterion let $\forall\vec{w} \in V_\alpha$ and assume that $\forall x, y, y' \in V_\alpha(\varphi^{V_\alpha}(x, y, \vec{w}) \wedge \varphi^{V_\alpha}(x, y', \vec{w}) \to y = y')$. Let $a \in V_\alpha$. Then

$$z = \{y | \exists x \in a\, \varphi^{V_\alpha}(x, y, \vec{w}))\} \cap V_\alpha$$

is a subset of V_α with $\mathrm{card}(z) \leqslant \mathrm{card}(a) < \alpha$. Hence $z \in V_\alpha$. □

Models of the form V_α can be used to show *relative consistencies*.

Theorem 4. *Let* ZF *be consistent. Then the theory consisting of all* ZFC*-axioms except Infinity together with the negation of Infinity is consistent.*

Proof. Assume that the theory consisting of all ZFC-axioms except Infinity together with the negation of Infinity is *inconsistent*, i.e. that it implies a contradiction like $\exists x\, x \not\equiv x$. ZF implies that the former theory holds in V_ω. So its implications hold in V_ω. Hence ZF implies $(\exists x\, x \not\equiv x)^{V_\omega} = \exists x \in V_\omega\, x \not\equiv x$. Thus ZF is inconsistent. □

The following lead Abraham Fraenkel to the introduction of the Replacement schema.

Theorem 5. *Let* Z *be the system of* Zermelo *set theory, consisting of the axioms of Extensionality, Pairing, Union, Power, Separation, Infinity, and Foundation. Then* Z *does not imply Replacement.*

Proof. (Sketch) $V_{\omega+\omega}$ is a model of Z but $V_{\omega+\omega}$ does not satisfy Replacement: define the map $F: \omega \to V_{\omega+\omega}$, $F(n) = V_{\omega+n}$. F is definable in $V_{\omega+\omega}$ by the $\in$-formula

$\varphi(x, y, \omega, V_\omega) = \exists f(f$ is a function $\wedge\ \mathrm{dom}(f) \in \omega \wedge x \in \mathrm{dom}(f) \wedge f(0) = V_\omega \wedge \forall n(n+1 \in \mathrm{dom}(f) \to \forall u(u \in f(n+1) \leftrightarrow u \subseteq f(n)))$.

φ formalises the definition of F by recursion on ω. Then $F[\omega] = \{V_{\omega+n} | n < \omega\} \notin V_{\omega+\omega}$, and so $V_{\omega+\omega}$ does not satisfy replacement for the formula φ. □

We shall discuss some details concerning the definition of φ inside W later.

Exercise 1. Define $H_\kappa = \{x | \mathrm{card}(\mathrm{TC}(\{x\})) < \kappa\}$. Examine which ZFC-axiom hold in H_κ for various κ.

3 Absoluteness and Reflection

In the study of models of set theory one passes from models $(W, \in)$ of set theory to other models $(W', \in)$, and one is interested in the behaviour of truth values of certain formulas. Some truth values are invariant or *absolute*.

Definition 6. *Let W, W' be terms and let $\varphi(x_0, \ldots, x_{n-1})$ be an $\in$-formula which does not have common variables with W or W'. φ is W-W'-*absolute *if*

$$\forall x_0, \ldots, x_{n-1} \in W \cap W' \ (\varphi^W \leftrightarrow \varphi^{W'}).$$

*If $W' = V$ we call φ W-*absolute*.*

In the next section we shall give syntactic criteria for absoluteness

Theorem 7. (Levy reflection theorem) *Assume* ZF*. Let $(W_\alpha)_{\alpha \in \mathrm{Ord}}$ be a* continuous hierarchy*, i.e.*

$$\alpha < \beta \rightarrow W_\alpha \subseteq W_\beta, \text{ and if } \lambda \text{ is a limit ordinal then } W_\lambda = \bigcup_{\alpha < \lambda} W_\alpha.$$

Let $W = \bigcup_{\alpha \in \mathrm{Ord}} W_\alpha$ be the limit of the hierarchy. Let $\varphi_0(\vec{x}), \ldots, \varphi_{n-1}(\vec{x})$ be a finite list of $\in$-formulas. Let $\theta_0 \in \mathrm{Ord}$. Then there exists a limit ordinal $\theta > \theta_0$ such that $\varphi_0(\vec{x}), \ldots, \varphi_{n-1}(\vec{x})$ are W_θ-W-absolute.

Proof. We may assume that the $\in$-formulas φ_i are only built using $\neg$, $\wedge$, $\exists$ and that all subformulas of φ_i occur in the initial part $\varphi_0(\vec{x}), \ldots, \varphi_{i-1}(\vec{x})$ of the list of formulas. Let r be the length of the vector $\vec{x}$. For $i < n$ define functions $F_i: W^r \rightarrow \mathrm{Ord}$ by

$$F_i(\vec{x}) = \begin{cases} \min\{\beta | \exists v \in W_\beta \, \psi^W(\vec{x})\}, \text{ if } \varphi_i = \exists v \psi \text{ and } \exists v \in W \, \psi^W(\vec{x}) \\ 0, \text{ else} \end{cases}$$

By the definition of F,

$$\forall \vec{x} \in W \ (\exists v \in W \, \psi^W(\vec{x}) \leftrightarrow \exists v \in W_{F_i(\vec{x})} \, \psi^W(\vec{x})). \tag{1}$$

Using the Replacement schema, recursively define an ω-sequence $(\theta_m)_{m < \omega}$ starting with the given θ_0 by

$$\theta_{m+1} = \bigcup \{F_i(\vec{x}) | i < n \wedge \vec{x} \in W_{\theta_m}\} \cup (\theta_m + 1).$$

Define the limit ordinal $\theta = \bigcup_{m<\omega} \theta_m$. Then for $\varphi_i = \exists v \psi$ from the list and $\vec{x} \in W_\theta$

$$\exists v \in W_\theta\, \psi^W(\vec{x}) \leftrightarrow \exists v \in W_{F_i(\vec{x})}\, \psi^W(\vec{x}). \tag{2}$$

Now we show by induction on $i < n$ that φ_i is W_θ-W-absolute. Let $\vec{x} \in W_\theta$.

Case 1. φ_i is atomic. Then φ_i is trivially absolute.

Case 2. $\varphi_i = \neg\varphi_j$ with $j < i$. Then $\varphi_i^{W_\theta}(\vec{x}) = \neg\varphi_j^{W_\theta}(\vec{x}) \leftrightarrow \neg\varphi_j^W(\vec{x}) = \varphi_i^W(\vec{x})$, using the induction hypothesis.

Case 3. $\varphi_i = \varphi_j \vee \varphi_k$ with $j, k < i$. Then $\varphi_i^{W_\theta}(\vec{x}) = \varphi_j^{W_\theta}(\vec{x}) \vee \varphi_k^{W_\theta}(\vec{x}) \leftrightarrow \varphi_j^W(\vec{x}) \vee \varphi_k^W(\vec{x}) = \varphi_i^W(\vec{x})$, using the induction hypothesis.

Case 4. $\varphi_i = \exists v \varphi_j$ with $j < i$. Then, using the induction hypothesis and (1) and (2)

$$\begin{array}{rcl}
\varphi_i^{W_\theta}(\vec{x}) & = & \exists v \in W_\theta\, \varphi_j^{W_\theta}(\vec{x}) \\
& \leftrightarrow & \exists v \in W_\theta\, \varphi_j^W(\vec{x}) \\
& \leftrightarrow & \exists v \in W_{F_i(\vec{x})}\, \varphi_j^W(\vec{x}) \\
& \leftrightarrow & \exists v \in W\, \varphi_j^W(\vec{x}) \\
& = & \varphi_i^W(\vec{x}).
\end{array}$$

□

Theorem 8. *If* ZF *is consistent then* ZF *is not equivalent to a finite system of axioms.*

Proof. Work in ZF. Assume for a contradiction that ZF is equivalent to the list $\varphi_0, \ldots, \varphi_{n-1}$ of formulas without free variables. By the reflection theorem, Theorem 7, there exists $\theta \in \mathrm{Ord}$ such that $\varphi_0^{V_\theta}, \ldots, \varphi_{n-1}^{V_\theta}$. Thus ZF implies

$$\exists w (w \text{ is transitive} \wedge \varphi_0^w \wedge \ldots \wedge \varphi_{n-1}^w). \tag{3}$$

By Foundation take an $\in$-minimal such w_0. Since the $\varphi_0, \ldots, \varphi_{n-1}$ imply all of ZF, they also imply (3). Therefore

$$(\exists w (w \text{ is transitive} \wedge \varphi_0^w \wedge \ldots \wedge \varphi_{n-1}^w))^{w_0}.$$

This is equivalent to

$$\exists w \in w_0 ((w \text{ is transitive})^{w_0} \wedge (\varphi_0^w)^{w_0} \wedge \ldots \wedge (\varphi_{n-1}^w)^{w_0}).$$

Let $w_1 \in w_0$ be such a w. Since w_0 is transitive, $w_1 \subseteq w_0$. Relativising to w_1 *and* to w_0 is equivalent to relativising to $w_1 \cap w_0 = w_1$:

$$(w_1 \text{ is transitive})^{w_0} \wedge \varphi_0^{w_1} \wedge \ldots \wedge \varphi_{n-1}^{w_1}.$$

Let "w_1 is transitive" be the formula

$$\forall u \in w_1 \forall v \in u\, v \in w_1\,.$$

This is equivalent to

$$\forall u \in w_1 \cap w_0 \forall v \in u \cap w_0\ v \in w_1$$

and to

$$(\forall u \in w_1 \forall v \in u\ v \in w_1)^{w_0}.$$

Hence

$$w_1 \text{ is transitive} \wedge \varphi_0^{w_1} \wedge \ldots \wedge \varphi_{n-1}^{w_1}.$$

This contradicts the $\in$-minimality of w_0. □

Similarly one gets

Theorem 9. *Let Φ be a collection of $\in$-formulas which is a consistent extension of the axiom system* ZF*. Then Φ is not finitely axiomatisable. So is* ZFC *is consistent it is not finitely axiomatisable.*

We can also use the reflection theorem to "justify" the assumption of transitive models of set theory.

Theorem 10. *Let* ZF *be consistent. Then the theory* ZF + M *is transitive*+ZF^M *is consistent where M is a new variable.*

Proof. Assume that ZF + M is transitive+ZF^M is inconsistent. Then the inconsistency follows from finitely many formulas of that theory. Take ZF-axioms $\varphi_0, \ldots, \varphi_{n-1}$ such that

$$\varphi_0, \ldots, \varphi_{n-1}, \varphi_0^M, \ldots, \varphi_{n-1}^M, M \text{ is transitive}$$

imply the inconsistent statement $x \neq x$. Work in ZF. By Reflection, Theorem 7, there is some V_θ such that $\varphi_0, \ldots, \varphi_{n-1}$ are V_θ-absolute. Then the following hold:

$$\varphi_0, \ldots, \varphi_{n-1}, \varphi_0^{V_\theta}, \ldots, \varphi_{n-1}^{V_\theta}, V_\theta \text{ is transitive.}$$

But then the proof of $x \neq x$ can be carried out under the assignment $M \mapsto V_\theta$. This means that ZF is inconsistent. □

Similarly:

Theorem 11. *Let* ZFC *be consistent. Then the theory* ZFC + M *is transitive*+ZFC^M *is consistent where* M *is a new variable.*

4 Formalisation of Formal Languages

We want to construct G ˚ 6del's model HOD which stands for the class of Hereditarily Ordinal Definable sets. HOD will be a model of the theory ZFC. The basic intuitions are:

- we want to define some "minimal" model of set theory which only contains "necessary".
- a model of set theory must be closed under definable sets where definitions may contain parameters from that model.
- one might define the model as the collection of all sets definable from parameters out of some reasonable class.
- one could take the class Ord of ordinals as the class of parameters: the class OD of Ordinal Definable sets is the collection of all sets of the form

  $$y = \{x | \varphi(x, \vec{\alpha})\}$$

  where φ is a formula of set theory and $\vec{\alpha} \in \mathrm{Ord}$.
- this may leed to a class which satisfies the axiom of choice since we can wellorder the collection of terms $\{x | \varphi(x, \vec{\alpha})\}$ by wellordering the countable set of formulas and the finite sequences of parameters.
- to get a transitive model we also need that elements $x \in y$ are also ordinal definable, that $u \in x \in y$ are ordinal definable etc., i.e. that y is *hereditarily* ordinal definable. That means $\mathrm{TC}(\{y\}) \subseteq \mathrm{OD}$.

So far we do not have a definition of HOD by a formula of set theory, since we are ranging over *all* formulas φ of set theory. This makes arguing about HOD in ZF difficult. G ˚ 6DEL's crucial observation is that HOD is, after all, definable by a single $\in$-formula which roughly is as follows:

$$z \in \mathrm{HOD} \leftrightarrow \mathrm{TC}(\{z\}) \subseteq \mathrm{OD}$$

and

$$y \in \mathrm{OD} \leftrightarrow \text{there exists an } \in\text{-formula } \varphi \text{ and } \vec{\alpha} \in \mathrm{Ord} \text{ such that } y = \{x | \varphi(x, \vec{\alpha})\}.$$

To turn the right-hand side into an $\in$-formula one has to formalise the collection of all $\in$-formulas in set theory and also the truth predicate $\varphi(x, \vec{\alpha})$ as a new formula in the variables φ (sic!), x, and $\vec{\alpha}$.

Consider the language of set theory formed by variables v_0, v_1, ..., the relations $\equiv$ and $\in$, and logical signs $\neg$, $\vee$, $\exists$. Code formulas φ of that language into sets $\ulcorner\varphi\urcorner$ by recursion on the structure of φ as follows.

Definition 12. *For a formula φ of set theory define the* G ˚ 6delisation $\ulcorner\varphi\urcorner$ *by recursion:*

- $\ulcorner v_i \equiv v_j \urcorner = (0, i, j)$
- $\ulcorner v_i \in v_j \urcorner = (1, i, j)$
- $\ulcorner \neg\varphi \urcorner = (2, \ulcorner\varphi\urcorner, \ulcorner\varphi\urcorner)$
- $\ulcorner \varphi \vee \psi \urcorner = (3, \ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)$
- $\ulcorner \exists v_i \varphi \urcorner = (4, i, \ulcorner\varphi\urcorner)$

Note that $\ulcorner\varphi\urcorner \in V_\omega$ since V_ω contains all the natural numbers and is closed unter ordered triples. Next define the collection Fml of all (formal) formulas.

Definition 13. *By recursion on the wellfounded relation*

$$yRx \leftrightarrow \exists u, v\ (x = (u, y, v) \vee x = (u, v, y))$$

define

$$\begin{aligned} x \in \mathrm{Fml} \ \leftrightarrow\ & \exists i, j < \omega\ x = (0, i, j) \\ & \vee \exists i, j < \omega\ x = (1, i, j) \\ & \vee \exists y\, (y \in \mathrm{Fml} \wedge x = (2, y, y)) \\ & \vee \exists y, z (y \in \mathrm{Fml} \wedge z \in \mathrm{Fml} \wedge x = (3, y, z)) \\ & \vee \exists i < \omega \exists y (y \in \mathrm{Fml} \wedge x = (4, i, y)). \end{aligned}$$

Fml is the *set* of formalised $\in$-formulas. We have: $\mathrm{Fml} \subseteq V_\omega$, and for every standard $\in$-formula φ:

$$\ulcorner\varphi\urcorner \in \mathrm{Fml}\,.$$

It is, however, possible that Fml contains *nonstandard* formulas which are not of the form $\ulcorner\varphi\urcorner$. One has to be very careful here since one is working in the vicinity of the G ˚ 6DEL incompleteness theorems.

We interpret elements of Fml in structures of the form (M, E) where E is a binary relation on the *set* M and in particular in models of the form $(M, \in)$ which is a short notation for the $\in$-relation restricted to M:

$$(M, \in) = (M, \{(u, v) | u \in M \wedge v \in M \wedge u \in v\}).$$

Definition 14. *Let* $\operatorname{Asn}(M) = {}^{<\omega}M = \{a | a\colon \operatorname{dom}(a) \to M, \exists n < \omega \operatorname{dom}(a) \subseteq n\ \}$ *be the set of* assignments *in M. We also denote the assignment a by* $a(0), \ldots, a(n-1)$ *in case that* $\operatorname{dom}(a) = n$. *For* $a \in \operatorname{Asn}(M)$, $x \in M$, *and* $i < \omega$ *define the modified assigment* $a\frac{x}{i}$ *by*

$$a\frac{x}{i}(m) = \begin{cases} a(m), \text{ if } m \neq i \\ x, \text{ else} \end{cases}$$

Definition 15. *For a structure* (M, E) *with* $M \in V$, $\varphi \in \operatorname{Fml}$, *and a an assignment in M define the* satisfaction relation $(M, E) \vDash \varphi[a]$ *("*(M, E) *is* a model of φ *under the assignment a") by recursion on the complexity of* φ*:*

- $(M, E) \vDash (0, i, j)[a]$ *iff* $a(i) = a(j)$
- $(M, E) \vDash (1, i, j)[a]$ *iff* $a(i) E a(j)$
- $(M, E) \vDash (2, y, y)[a]$ *iff not* $(M, E) \vDash y[a]$
- $(M, E) \vDash (3, y, z)[a]$ *iff* $(M, E) \vDash y[a]$ *or* $(M, E) \vDash z[a]$
- $(M, E) \vDash (4, i, y)[a]$ *iff there exists* $x \in M$*:* $(M, E) \vDash y[a\frac{x}{i}]$

If $\operatorname{dom}(a) = n$ *we also write* $(M, E) \vDash \varphi[a(0), \ldots, a(n-1)]$.

Note that the recursion requires that M is a *set* since in the last clause we recurse to $(M, E) \vDash y[a\frac{x}{i}]$ for $x \in M$ and we cannot recurse to a proper class of preconditions.

The satisfaction relation agrees with the notion of "model" in terms of relativisations. A straightforward induction on the complexity of formulas shows:

Lemma 16. *Let* $\varphi(v_0, \ldots, v_{n-1})$ *an* $\in$*-formula. Then for any set M with* $a \in M$

$$\forall v_0, \ldots, v_{n-1} \in M((M, \in) \vDash \ulcorner \varphi \urcorner [v_0, \ldots, v_{n-1}] \leftrightarrow \varphi^M).$$

Exercise 2. Define a wellorder $<_{\operatorname{Fml}}$ of the set Fml in ordertype ω without using parameters.

Exercise 3. Show: for any $\varphi \in \operatorname{Fml}$ there is $n < \omega$ such that for any structure (M, E) and assignments b, b' in M:

$$\text{if } b \upharpoonright n = b' \upharpoonright n \text{ then } ((M, E) \vDash \varphi[b] \leftrightarrow (M, E) \vDash \varphi[b']).$$

5 Heriditarily Ordinal Definable Sets

We can now give the (official) definition of the class HOD.

Definition 17. *Define*

$$\mathrm{OD} = \{y | \exists \alpha \in \mathrm{Ord}\, \exists \varphi \in \mathrm{Fml}\, \exists a \in \mathrm{Asn}(\alpha)\;\; y = \{z \in V_\alpha | (V_\alpha, \in) \vDash \varphi[a\frac{z}{0}]\}\},$$

and

$$\mathrm{HOD} = \{x | \mathrm{TC}(\{x\}) \subseteq \mathrm{OD}\}$$

We shall see that HOD is a model of ZFC.

Lemma 18. $\mathrm{Ord} \subseteq \mathrm{OD}$ *and* $\mathrm{Ord} \subseteq \mathrm{HOD}$.

Proof. Let $\xi \in \mathrm{Ord}$. Then

$$\begin{aligned} \xi &= \{z \in V_{\xi+1} | z \in \xi\} \\ &= \{z \in V_{\xi+1} | (z \in \xi)^{V_{\xi+1}}\} \\ &= \{z \in V_{\xi+1} | (V_{\xi+1}, \in) \vDash \ulcorner v_0 \in v_1 \urcorner [z, \xi]\} \\ &\in \mathrm{OD} \end{aligned}$$

If $\xi \in \mathrm{Ord}$ then $\mathrm{TC}(\{\xi\}) = \xi + 1 \subseteq \mathrm{OD}$ and so $\xi \in \mathrm{HOD}$. $\square$

Lemma 19. HOD *is transitive.*

Proof. Let $x \in y \in \mathrm{HOD}$. Then $\mathrm{TC}(\{x\}) \subseteq \mathrm{TC}(\{y\}) \subseteq \mathrm{OD}$ and so $x \in \mathrm{HOD}$. $\square$

An element $y = \{z \in V_\alpha | (V_\alpha, \in) \vDash \varphi[a\frac{z}{0}]\}$ of OD is determined or *named* by the tripel (V_α, φ, a).

Definition 20. *For* $x \in V$, $\varphi \in \mathrm{Fml}$, *and* $a \in \mathrm{Asn}(x)$ *define the* interpretation function

$$I(x, \varphi, a) = \{z \in x | (x, \in) \vDash \varphi[a\frac{z}{0}]\}.$$

We say that $I(x, \varphi, a)$ *is the* interpretation *of* (x, φ, a), *or that* (x, φ, a) *is a* name *for* $I(x, \varphi, a)$.

Lemma 21. *Let*

$$\mathrm{OD}^* = \{(V_\alpha, \varphi, a) | \alpha \in \mathrm{Ord}, \varphi \in \mathrm{Fml}, a \in \mathrm{Asn}(\alpha)\}$$

be the class of OD-names. *Then* $\mathrm{OD} = I[\mathrm{OD}^*]$. OD^* *has a wellorder* $<_{\mathrm{OD}^*}$ *of type* Ord *which is definable without parameters.*

Proof. Let $<_{\mathrm{Fml}}$ be a wellorder of Fml in ordertype ω which is definable without parameters (see Exercise 2).

Wellorder the class $\bigcup_{\alpha\in\mathrm{Ord}}\mathrm{Asn}(\alpha)$ of all relevant assignment by

$$\begin{aligned}a <_{\mathrm{Asn}} a' \ \leftrightarrow\ & \max(\mathrm{ran}(a)) < \max(\mathrm{ran}(a')) \\ & \vee (\max(\mathrm{ran}(a)) = \max(\mathrm{ran}(a')) \wedge \exists n \in \mathrm{dom}(a')(a \upharpoonright n = a' \upharpoonright n \wedge (n \notin \mathrm{dom}(a) \vee \\ & (n \in \mathrm{dom}(a) \wedge a(n) < a'(n)))))\end{aligned}$$

Wellorder OD^* in ordertype Ord by

$$\begin{aligned}(V_\alpha, \varphi, a) <_{\mathrm{OD}^*} (V_{\alpha'}, \varphi', a') \ \leftrightarrow\ & \alpha < \alpha' \\ & \vee(\alpha = \alpha' \wedge \varphi <_{\mathrm{Fml}} \varphi') \\ & \vee(\alpha = \alpha' \wedge \varphi = \varphi' \wedge a <_{\mathrm{Asn}} a').\end{aligned}$$

□

Lemma 22. OD *has a wellorder* $<_{\mathrm{OD}}$ *of type* Ord *which is definable without parameters.*

Proof. We let $<_{\mathrm{OD}}$ be the wellorder induced by $<_{\mathrm{OD}^*}$ via I:

$$\begin{aligned}x <_{\mathrm{OD}} x' \ \leftrightarrow\ & \exists(V_\alpha, \varphi, a) \in \mathrm{OD}^*(x = I(V_\alpha, \varphi, a) \wedge \forall(V_{\alpha'}, \varphi', a') \in \mathrm{OD}^*(x' = I(V_{\alpha'}, \varphi', a') \rightarrow \\ & (V_\alpha, \varphi, a) <_{\mathrm{OD}^*} (V_{\alpha'}, \varphi', a'))).\end{aligned}$$

□

Lemma 23. *Let z be definable from $x_1, \ldots, x_{n-1}$ by the $\in$-formula $\varphi(v_1, \ldots, v_n)$:*

$$\forall v_n(v_n = z \leftrightarrow \varphi(x_1, \ldots, x_{n-1}, v_n)). \tag{4}$$

Let $x_1, \ldots, x_n \in \mathrm{OD}$ and $z \subseteq \mathrm{HOD}$. Then $z \in \mathrm{HOD}$.

Proof. $\mathrm{TC}(\{z\}) = \{z\} \cup \mathrm{TC}(z) \subseteq \{z\} \cup \mathrm{HOD}$. So it suffices to prove $z \in \mathrm{OD}$. Using the canonical wellorder $<_{\mathrm{OD}}$ from Lemma 22 every element x of OD is definable from one ordinal δ without further parameters: x is the δ-th element in the wellorder $<_{\mathrm{OD}}$. So we may simply assume that the parameters $x_1, \ldots, x_{n-1}$ *are* ordinals.

Let $z, x_1, \ldots, x_{n-1} \in V_{\theta_0}$. By Reflection take some $\theta > \theta_0$ such that φ is V_θ-absolute. Then

$$\begin{aligned}z &= \{u \in V_\theta | u \in z\} \\ &= \{u \in V_\theta | \exists v_n (\varphi(x_1, \ldots, x_{n-1}, v_n) \wedge u \in v_n)\} \\ &= \{u \in V_\theta | \exists v_n \in V_\theta (\varphi(x_1, \ldots, x_{n-1}, v_n)^{V_\theta} \wedge u \in v_n)\} \\ &= \{u \in V_\theta | (V_\theta, \in) \vDash \ulcorner \exists v_n (\varphi(v_1, \ldots, v_{n-1}, v_n) \wedge v_0 \in v_n) \urcorner [u, x_1, \ldots, x_{n-1}]\} \\ &\in \mathrm{OD}.\end{aligned}$$

□

The two previous Lemmas justify the notion "ordinal definable": if $z \in \mathrm{OD}$ it is definable as the δ-th element in $<_{\mathrm{OD}}$ for some ordinal δ. Conversely, if z is definable from ordinal parameters the preceding proof shows that $z \in \mathrm{OD}$.

Theorem 24. $\mathrm{ZF}^{\mathrm{HOD}}$.

Proof. Using the criteria of Theorem 2 we check certain closure properties of HOD.

a) Extensionality holds in HOD, since HOD is transitive.

b) Let $x, y \in \mathrm{HOD}$. Then $\{x, y\}$ is definable from x, y, and $\{x, y\} \subseteq \mathrm{HOD}$. By Lemma 23, $\{x, y\} \in \mathrm{HOD}$, i.e. HOD is closed with respect to unordered pairs. This implies Pairing in HOD.

c) Let $x \in \mathrm{HOD}$. Then $\bigcup x$ is definable from x, and $\bigcup x \subseteq \mathrm{TC}(\{x\}) \subseteq \mathrm{HOD}$. So $\bigcup x \in \mathrm{HOD}$, and so Union holds in HOD.

d) Let $x \in \mathrm{HOD}$. Then $\mathcal{P}(x) \cap \mathrm{HOD}$ is definable from x, and $\mathcal{P}(x) \cap \mathrm{HOD} \subseteq \mathrm{HOD}$. So $\mathcal{P}(x) \cap \mathrm{HOD} \in \mathrm{HOD}$ and Powerset holds in HOD.

e) $\omega \in \mathrm{HOD}$ implies that Infinity holds in HOD.

f) Let $\varphi(x, \vec{w})$ be an $\in$-formula and $\vec{w}, a \in \mathrm{HOD}$. Then $\{x \in a | \varphi^{\mathrm{HOD}}(x, \vec{w})\}$ is a set by Separation in V, and it is definable from $\vec{w}, a$. Moreover $\{x \in a | \varphi^{\mathrm{HOD}}(x, \vec{w})\} \subseteq \mathrm{HOD}$. So $\{x \in a | \varphi^{\mathrm{HOD}}(x, \vec{w})\} \in \mathrm{HOD}$, and Separation for the formula φ holds in HOD.

g) Let $\varphi(x, y, \vec{w})$ be an $\in$-formula and $\vec{w}, a \in \mathrm{HOD}$. Assume that

$$\forall x, y, y' \in \mathrm{HOD}(\varphi^{\mathrm{HOD}}(x, y, \vec{w}) \wedge \varphi^{\mathrm{HOD}}(x, y', \vec{w}) \rightarrow y = y').$$

Then $\{y | \exists x \in a \varphi^{\mathrm{HOD}}(x, y, \vec{w})\} \cap \mathrm{HOD}$ is a set by Replacement and Separation in V. It is definable from $\vec{w}$, a. Moreover $\{y | \exists x \in a \varphi^{\mathrm{HOD}}(x, y, \vec{w})\} \cap \mathrm{HOD} \subseteq \mathrm{HOD}$. So $\{y | \exists x \in a \varphi^{\mathrm{HOD}}(x, y, \vec{w})\} \cap \mathrm{HOD} \in \mathrm{HOD}$, and Replacement for φ holds in HOD.

h) Foundation holds in HOD since HOD is an $\in$-model. $\square$

Hence HOD is an *inner model of set theory*, i.e. HOD is transitive, contains all ordinals, and is a model of ZF.

Theorem 25. $\mathrm{AC}^{\mathrm{HOD}}$.

Proof. We prove AC in HOD using Theorem 2. Consider $x \in \mathrm{HOD}$ with $\emptyset \notin x \wedge \forall u, u' \in x\, (u \neq u' \rightarrow u \cap u' = \emptyset)$. Define a choice set y for x by

$$y = \{v | \exists u \in x \colon v \text{ is the } <_{\mathrm{OD}}\text{-minimal element of } u\}.$$

Obviously y intersects every element of x in exactly one element. y is definable from $x \in \mathrm{HOD}$ and $y \subseteq \mathrm{HOD}$. By Lemma 23, $y \in \mathrm{HOD}$, as required. $\square$

Theorem 26. (Kurt G˚6del, 1938) *If* ZF *is consistent then* ZFC *is consistent. In other words: the Axiom of Choice is* relatively consistent *with the system* ZF.

Proof. Since ZF proves that HOD is a model for ZFC. □

Exercise 4. Extend the formal language by atomic formulas for “$x \in A$” where A is considered a unary predicate or relation. Define

$$\mathrm{OD}(A) = \{y \mid \exists \alpha \in \mathrm{Ord}\, \exists \varphi \in \mathrm{Fml}'\, \exists \beta \colon \omega \to A \cap V_\alpha \ y = \{z \in V_\alpha \mid (V_\alpha, A \cap V_\alpha, \in) \vDash \varphi[\beta \frac{z}{0}]\}\}$$

and the corresponding generalisation $\mathrm{HOD}(A)$ of HOD. Prove:

a) if A is transitive then $A \subseteq \mathrm{HOD}(A)$;

b) if A is moreover definable from some parameters $a_0, \ldots, a_{n-1} \in A$ then $\mathrm{ZF}^{\mathrm{HOD}(A)}$.

Note that AC does in general not hold in $\mathrm{HOD}(A)$.

6 Absolute and Definite Notions

For terms we define:

Definition 27. *Let W be a term, and $t(\vec{x}) = \{y \mid \varphi(y, \vec{x})\}$ be a term which has no common variables with W. Define the* relativisation

$$t^W(\vec{x}) = \{y \in W \mid \varphi^W(y, \vec{x})\}.$$

*Let W' be another term which has no common variables with t. Then t is W-W'-*absolute *if*

$$\forall \vec{x} \in W \cap W'((t^W(\vec{x}) \in W \leftrightarrow t^{W'}(\vec{x}) \in W') \wedge (t^W(\vec{x}) \in W \to t^W(\vec{x}) = t^{W'}(\vec{x}))).$$

*If $W' = V$ we call t W-*absolute.

Formulas and terms may be absolute for complicated reasons. In this section we want to study notions that are absolute between *all* transitive models of ZF^- simply due to their syntactical structure.

Definition 28. *Let $\psi(\vec{v})$ be an $\in$-formula and let $t(\vec{v})$ be a term, both in the free variables $\vec{v}$. Then*

a) ψ is definite *iff for every transitive ZF^--model $(M, \in)$*

$$\forall \vec{x} \in M\ (\psi^M(\vec{x}) \leftrightarrow \psi(\vec{x})).$$

b) t is definite *iff for every transitive* ZF$^-$*-model* $(M,\in)$

$$\forall\vec{x}\in M\, t^M(\vec{x})\in M \textit{ and } \forall\vec{x}\in M\, t^M(\vec{x})=t(\vec{x}).$$

We shall prove that most simple set-theoretical notions are definite. We shall work inductively: basic notions are definite and important set-theoretical operations lead from definite notions to definite notions.

The following lemma shows that the operations of relativisation and substitution of a term into a formula commute.

Lemma 29. *Let $\varphi(x,\ \vec{y})$ be a formula, $t(\vec{z})$ be a term, and M be a class. Assume that $\forall\vec{z}\in M\, t(\vec{z})\in M$. Then*

$$\forall\vec{y},\vec{z}\in M\ (\varphi(t(\vec{z}),\vec{y}))^M\leftrightarrow\varphi^M(t^M(\vec{z}),\vec{y})).$$

Proof. If $t=t(\vec{z})$ is of the form $t=z$ then there is nothing to show. Assume otherwise that t is of the form $t=\{u|\psi(u,\ \vec{z})\}$. We work by induction on the complexity of φ. Assume that $\varphi\equiv x=y$ and $y,\vec{z}\in M$. Then

$$\begin{aligned}
(t(\vec{z})=y)^M &\leftrightarrow (\{u|\psi(u,\vec{z})\}=y)^M\\
&\leftrightarrow (\forall u\ (\psi(u,\vec{z})\leftrightarrow u\in y))^M\\
&\leftrightarrow \forall u\in M\ (\psi^M(u,\vec{z})\leftrightarrow u\in y)\\
&\leftrightarrow \{u\in M|\psi^M(u,\vec{z})\}=y\\
&\leftrightarrow t^M(\vec{z})=y\\
&\leftrightarrow \varphi^M(t^M(\vec{z}),y)
\end{aligned}$$

Assume that $\varphi\equiv y\in x$ and $y,\vec{z}\in M$. Then

$$\begin{aligned}
(y\in t(\vec{z}))^M &\leftrightarrow \psi^M(\frac{y}{u},\vec{z})\\
&\leftrightarrow y\in\{u\in M|\psi^M(u,\vec{z})\}\\
&\leftrightarrow y\in t^M(\vec{z})\\
&\leftrightarrow \varphi^M(t^M(\vec{z}),y)
\end{aligned}$$

Assume that $\varphi\equiv x\in y$ and $y,\vec{z}\in M$. Then

$$\begin{aligned}
(t(\vec{z})\in y)^M &\leftrightarrow (\exists u\,(u=t(\vec{z})\wedge u\in y)^M\\
&\leftrightarrow \exists u\in M\,((u=t(\vec{z}))^M\wedge u\in y)\\
&\leftrightarrow \exists u\in M\,(u=t^M(\vec{z})\wedge u\in y), \text{ by the first case,}\\
&\leftrightarrow \exists u\,(u=t^M(\vec{z})\wedge u\in y), \text{ since } M \text{ is closed w.r.t. } t,\\
&\leftrightarrow t^M(\vec{z})\in y\\
&\leftrightarrow \varphi^M(t^M(\vec{z}),y)
\end{aligned}$$

The induction steps are obvious since the terms t resp. t^M are only substituted into the atomic subformulas of φ. $\square$

Theorem 30.

a) *The formulas $x = y$ and $x \in y$ are definite.*

b) *If the formulas φ and ψ are definite then so are $\neg\varphi$ and $\varphi \vee \psi$.*

c) *Let the formula $\varphi(x, \vec{y})$ and the term $t(\vec{z})$ be definite. Then so are $\varphi(t(\vec{z}), \vec{y})$ and $\exists x \in t(\vec{z})\ \varphi(x, \vec{y})$.*

d) *The terms x, $\emptyset$, $\{x, y\}$, and $\bigcup x$ are definite.*

e) *Let the terms $t(x, \vec{y})$ and $r(\vec{z})$ be definite. Then so is $t(r(\vec{z}), \vec{y})$.*

f) *Let the formula $\varphi(x, \vec{y})$ be definite. Then so is the term $\{x \in z \,|\, \varphi(x, \vec{y})\}$.*

g) *Let the term $t(x, \vec{y})$ be definite. Then so is the term $\{t(x, \vec{y}) \,|\, x \in z\}$.*

h) *The formulas "R is a relation", "f is a function", "f is injective", and "f is surjective" are definite.*

i) *The formulas $\mathrm{Trans}(x)$, $\mathrm{Ord}(x)$, $\mathrm{Succ}(x)$, and $\mathrm{Lim}(x)$ are definite.*

j) *The term ω is definite.*

Proof. Let M be a transitive ZF^--model.
a) is obvious since $(x = y)^M \equiv (x = y)$ and $(x \in y)^M \equiv (x \in y)$.
b) Assume that φ and ψ are definite and that $(M, \in)$ is a transitive ZF^--model. Then $\forall \vec{x} \in M\ (\varphi^M(\vec{x}) \leftrightarrow \varphi(\vec{x}))$ and $\forall \vec{x} \in M\ (\psi^M(\vec{x}) \leftrightarrow \psi(\vec{x}))$. Thus

$$\forall \vec{x} \in M\ ((\varphi \vee \psi)^M(\vec{x}) \leftrightarrow (\varphi^M(\vec{x}) \vee \psi^M(\vec{x})) \leftrightarrow (\varphi(\vec{x}) \vee \psi(\vec{x})) \leftrightarrow (\varphi \vee \psi)(\vec{x}))$$

and

$$\forall \vec{x} \in M\ ((\neg\varphi(\vec{x}))^M \leftrightarrow \neg(\varphi^M(\vec{x})) \leftrightarrow \neg(\varphi(\vec{x})) \leftrightarrow (\neg\varphi)(\vec{x})).$$

c) Let $(M, \in)$ be a transitive ZF^--model. Let $\vec{y}, \vec{z} \in M$. $t(\vec{z}) \in M$ since t is definite. Then

$$\begin{aligned}(\varphi(t(\vec{z}), \vec{y}))^M &\leftrightarrow \varphi^M(t^M(\vec{z}), \vec{y}), \text{ by Lemma 29},\\ &\leftrightarrow \varphi^M(t(\vec{z}), \vec{y}), \text{ since } t \text{ is definite},\\ &\leftrightarrow \varphi(t(\vec{z}), \vec{y}), \text{ since } \varphi \text{ is definite}.\end{aligned}$$

Also

$$
\begin{aligned}
(\exists x \in t(\vec{z})\ \varphi(x,\vec{y}))^M &\leftrightarrow (\forall x\,(x \in t(\vec{z}) \to \varphi(x,\vec{y})))^M \\
&\leftrightarrow \forall x \in M\,((x \in t(\vec{z}))^M \to \varphi^M(x,\vec{y})) \\
&\leftrightarrow \forall x \in M\,(x \in t^M(\vec{z}) \to \varphi^M(x,\vec{y})) \\
&\leftrightarrow \forall x \in M\,(x \in t(\vec{z}) \to \varphi(x,\vec{y})),\text{ since } t \text{ and } \varphi \text{ are definite,} \\
&\leftrightarrow \forall x\,(x \in t(\vec{z}) \to \varphi(x,\vec{y})),\text{ since } t(\vec{z}) \subseteq M, \\
&\leftrightarrow \forall x \in t(\vec{z})\ \varphi(x,\vec{y})).
\end{aligned}
$$

d) A variable term x is trivially definite, since $x^M = x$.

Consider the term $\emptyset = \{u | u \neq u\}$. Since M is non-empty and transitive, $\emptyset \in M$. Also

$$\emptyset^M = \{u \in M | u \neq u\} = \emptyset.$$

Consider the term $\{x, y\}$. For $x, y \in M$:

$$\{x,y\}^M = \{u \in M\,|\, u = x \vee u = y\} = \{u | u = x \vee u = y\} = \{x,y\}.$$

The pairing axiom in M states that

$$(\forall x, y \exists z\ z = \{x,y\})^M.$$

This implies

$$\forall x, y \in M \exists z \in M\, z = \{x,y\}^M = \{x,y\}$$

and

$$\forall x, y \in M\ \ \{x,y\} \in M.$$

Consider the term $\bigcup x$. For $x \in M$:

$$(\bigcup x)^M = \{u \in M | (\exists v \in x\ u \in v)^M\} = \{u \in M | \exists v \in x \cap M\ u \in v\} = \{u | \exists v \in x\ u \in v\} = \bigcup x.$$

The union axiom in M states that

$$(\forall x \exists z\ z = \bigcup x)^M.$$

This implies

$$\forall x \in M \exists z \in M\ z = (\bigcup x)^M = \bigcup x$$

and

$$\forall x \in M \bigcup x \in M.$$

$e)$ is obvious.

$f)$ Let $\vec{y}, z \in M$. By the separation schema in M,

$$(\exists w\ w = \{x \in z \,|\, \varphi(x, \vec{y})\})^M,$$

i.e. $\{x \in z \,|\, \varphi(x, \vec{y})\}^M \in M$. Moreover by the definiteness of φ

$$\{x \in z \,|\, \varphi(x, \vec{y})\}^M = \{x \in M \,|\, x \in y \wedge \varphi^M(x, \vec{y})\} = \{x \,|\, x \in y \wedge \varphi(x, \vec{y})\} = \{x \in z \,|\, \varphi(x, \vec{y})\}.$$

$g)$ Since t is definite, $\forall x, \vec{y} \in M\, t^M(x, \vec{y}) \in M$. This implies

$$\forall x, \vec{y} \in M \exists w \in M\, w = t^M(x, \vec{y})$$

and $(\forall x, \vec{y} \exists w\ w = t(x, \vec{y}))^M$. Let $\vec{y}, z \in M$. By replacement in M,

$$(\exists a\ a = \{t(x, \vec{y}) | x \in z\})^M.$$

Hence $\{t(x, \vec{y}) | x \in z\}^M \in M$. Moreover

$$\begin{aligned}
\{t(x, \vec{y}) | x \in z\}^M &= \{w | \exists x \in z\, w = t(x, \vec{y})\}^M \\
&= \{w \in M \,|\, \exists x \in z\, w = t^M(x, \vec{y})\} \\
&= \{w \,|\, \exists x \in z\, w = t^M(x, \vec{y})\}, \text{ since } M \text{ is closed w.r.t. } t^M, \\
&= \{w \,|\, \exists x \in z\, w = t(x, \vec{y})\}, \text{ since } t \text{ is definite}, \\
&= \{t(x, \vec{y}) | x \in z\}.
\end{aligned}$$

$h)$ "R is a relation" is equivalent to

$$\forall z \in R \exists x, y \in (\bigcup \bigcup z)\ z = \{\{x\}, \{x, y\}\}.$$

This is definite, using $c), d), e)$. The other relational statements are definite for similar reasons.

$i)$

$$\begin{aligned}
\mathrm{Trans}(x) &\leftrightarrow \forall y \in x \forall z \in y\, z \in x \\
\mathrm{Ord}(x) &\leftrightarrow \mathrm{Trans}(x) \wedge \forall y \in x\ \mathrm{Trans}(y) \\
\mathrm{Succ}(x) &\leftrightarrow \mathrm{Ord}(x) \wedge \exists y \in x\, x = y \cup \{y\} \\
\mathrm{Lim}(x) &\leftrightarrow \mathrm{Ord}(x) \wedge \neg\mathrm{Succ}(x) \wedge x \neq \emptyset
\end{aligned}$$

$j)$ Consider the term $\omega = \bigcap \{x | x \text{ is inductive}\}$. Since M satisfies the axiom of infinity,

$$\exists x \in M\ (x = \omega)^M.$$

Take $x_0 \in M$ such that $(x_0 = \omega)^M$. Then $(\mathrm{Lim}(x_0))^M$, $(\forall y \in x_0 \neg \mathrm{Lim}(y))^M$. By definiteness, $\mathrm{Lim}(x_0)$, $\forall y \in x_0 \neg \mathrm{Lim}(y)$, i.e. x_0 is equal to the smallest limit ordinal ω. Hence $\omega \in M$. The formula "x is inductive" has the form

$$\emptyset \in x \wedge \forall y \in x \bigcup \{y, \{y\}\} \in x$$

and is definite by previous considerations. Now

$$\begin{array}{rcl}
\omega^M & = & (\bigcap \{x | x \text{ is inductive}\})^M \\
& = & (\{y | \forall x\, (x \text{ is inductive} \rightarrow y \in x)\})^M \\
& = & \{y \in M | \forall x \in M\, (x \text{ is inductive} \rightarrow y \in x)\}, \text{ since "}x\text{ is inductive" is definite,} \\
& = & \bigcap \{x \in M | x \text{ is inductive}\} \\
& = & \bigcap \{x \cap \omega | x \in M \text{ is inductive}\}, \text{ since } \omega \in M, \\
& = & \bigcap \{\omega\}, \text{ since } \omega \text{ is the smallest inductive set,} \\
& = & \omega.
\end{array}$$

□

We may view this theorem as a "definite" form of the ZF$^-$-axioms: common notions and terms of set theory and mathematics are definite, and natural operations lead to further definite terms. Since the recursion principle is so important, we shall need a definite recursion schema:

Theorem 31. *Let $G(w, \vec{y})$ be a definite term, and let $F(\alpha, \vec{y})$ be the canonical term defined by $\in$-recursion with G:*

$$\forall x\, F(x, \vec{y}) = G(\{(z, F(z, \vec{y})) | z \in x\}, \vec{y}).$$

Then the term $F(x, \vec{y})$ is definite.

Proof. Let M be a transitive ZF$^-$-model. By the recursion theorem, F is a total function in V and in M:

$$\forall x, \vec{y} \in M\, F^M(x, \vec{y}) \in M.$$

Assume that x were $\in$-minimal such that $F^M(x, \vec{y}) \neq F(x, \vec{y})$. Then we get a contradiction by

$$\begin{array}{rcl}
F^M(x, \vec{y}) & = & G^M(\{(z, F^M(z, \vec{y})) | z \in x\}, \vec{y}) \\
& = & G^M(\{(z, F(z, \vec{y})) | z \in x\}, \vec{y}), \text{ by the minimality of } x, \\
& = & G(\{(z, F(z, \vec{y})) | z \in x\}, \vec{y}), \text{ by the definiteness of } G, \\
& = & F(x, \vec{y}).
\end{array}$$

□

Lemma 32. rank(x) *is a definite term.*

Proof. $\mathrm{rank}(x) = \bigcup \{\mathrm{rank}(y) + 1 \,|\, y \in x\} = G(\mathrm{rank} \upharpoonright x)$ with the definite recursion rule

$$G(f) = \{f(z) + 1 \,|\, z \in \mathrm{dom}(f)\}$$

□

Theorem 33. *Let $G(w,\, \vec{y})$ be a definite term and let $R(z,\, x)$ be a strongly wellfounded relation such that the term $\{z | zRx\}$ is definite. Let $F(\alpha,\, \vec{y})$ be the canonical term defined by R-recursion with G:*

$$\forall x\, F(x, \vec{y}) = G(\{(z, F(z, \vec{y}))|zRx\}, \vec{y}).$$

Then the term $F(x, \vec{y})$ is definite.

Proof. Let M be a transitive ZF^--model. By the recursion theorem, F is a total function in V and in M:

$$\forall x, \vec{y} \in M\, F^M(x, \vec{y}) \in M.$$

Assume that x were R-minimal such that $F^M(x, \vec{y}) \neq F(x, \vec{y})$. Then we get a contradiction by

$$\begin{aligned} F^M(x, \vec{y}) &= G^M(\{(z, F^M(z, \vec{y}))|(zRx)^M\}, \vec{y}) \\ &= G^M(\{(z, F^M(z, \vec{y}))|zRx\}, \vec{y}), \text{ by the assumptions on } R, \\ &= G^M(\{(z, F(z, \vec{y}))|zRx\}, \vec{y}), \text{ by the minimality of } x, \\ &= G(\{(z, F(z, \vec{y}))|zRx\}, \vec{y}), \text{ by the definiteness of } G, \\ &= F(x, \vec{y}). \end{aligned}$$

□

Also other kinds of recursions lead from definite recursion rules to definite functions.

Note that *not every* important notion is definite. For the powerset operation we have $\mathcal{P}^M(x) = \mathcal{P}(x) \cap M$. If M does not contain all subsets of x then $\mathcal{P}^M(x) \neq \mathcal{P}(x)$. We shall later produce countable transitive models M of ZF^- so that $\mathcal{P}^M(\omega) \neq \mathcal{P}(\omega)$, and we thus prove that $\mathcal{P}(x)$ is *not* definite. Obviously the construction of models of set theory is especially geared at exhibiting the indefiniteness of particular notions.

Exercise 5. Show that (x, y), $x \times y$, $f \upharpoonright x$ are definite terms.

Exercise 6. Show that $\mathrm{TC}(x)$ is a definite term.

Exercise 7. Show that the term V_n for $n < \omega$ is definite. Show that the term V_ω is definite.

Lemma 34. *The following modeltheoretic notions are definite:*

a) *the term* Fml *of all formalised* $\in$*-formulas;*

b) *the term* $\mathrm{Asn}(M)$*;*

c) *the formula "*$(M,E)\vDash\varphi[b]$*" in the variables* M,E,φ,b.

Proof. $a)$ and $c)$. Fml and $\vDash$are defined by recursion on the relation

$$yRx\leftrightarrow\exists u,v\ (x=(u,y,v)\vee x=(u,v,y)).$$

Then

$$\{y|yRx\}=\{y\in\mathrm{TC}(x)|\exists u,v\in\mathrm{TC}(x)\ (x=(u,y,v)\vee x=(u,v,y))\}$$

is definite. Therefore the characteristic function of Fml is definite as well as the term

$$\mathrm{Fml}=\{x\in V_\omega|x\in\mathrm{Fml}\}.$$

By Theorem 33 on definite recursions, Fml and $\vDash$ are definite.

$b)$ Define by definite recursion $\mathrm{Asn}_0(M)=\{\emptyset\}$ and

$$\mathrm{Asn}_{n+1}(M)=\mathrm{Asn}_n(M)\cup\{a\frac{x}{n}\,|\,a\in\mathrm{Asn}_n(M)\wedge x\in M\}.$$

$\mathrm{Asn}_n(M)$ is a definite term, and $\mathrm{Asn}(M)=\bigcup\{\mathrm{Asn}_n(M)|n\in\omega\}$ is also definite. $\square$

7 Skolem hulls

Theorem 35. (Downward L˚6wenheim-Skolem Theorem, ZFC) *Let* $X\subseteq M\neq\emptyset$ *be sets. Then there exists* $N\subseteq M$ *such that*

a) $X\subseteq N$ *and* $\mathrm{card}(N)\leqslant\mathrm{card}(X)+\aleph_0$*;*

b) *every* $\in$*-formula is* N*-*M*-absolute.*

Proof. Take a wellorder $\prec$ of M. Define a Skolem *function* $S\colon\mathrm{Fml}\times\mathrm{Asn}(M)$,

$$S(\varphi,a)=\begin{cases}\text{the }\prec\text{ smallest element of }I(M,\varphi,a)\text{, if this exists,}\\ m_0\text{, else,}\end{cases}$$

where m_0 is some fixed element of M. Intuitively, $S(\varphi, a_0, a_1, \ldots, a_{k-1})$ is the $\prec$-smallest element $z \in M$ such that $M \vDash \varphi(z, a_1, \ldots, a_{k-1})$, if such a z exists.

Define $N_0 = X$, $N_1, N_2, \ldots$ recursively:

$$N_{n+1} = N_n \cup S[\mathrm{Fml} \times \mathrm{Asn}(N_n)],$$

and let $N = \bigcup_{n<\omega} N_n$.

We show inductively that $\mathrm{card}(N_n) \leqslant \mathrm{card}(X) + \aleph_0$:

$$\begin{aligned}
\mathrm{card}(N_{n+1}) &\leqslant \mathrm{card}(N_n) + \mathrm{card}(\mathrm{Fml} \times \mathrm{Asn}(N_n)) \\
&\leqslant \mathrm{card}(N_n) + \mathrm{card}(\mathrm{Fml}) \cdot \mathrm{card}({}^{<\omega}N_n) \\
&\leqslant \mathrm{card}(N_n) + \aleph_0 \cdot \mathrm{card}(N_n)^{<\omega} \\
&\leqslant \mathrm{card}(X) + \aleph_0 + \aleph_0 \cdot (\mathrm{card}(X) + \aleph_0), \text{ by inductive assumption,} \\
&\leqslant \mathrm{card}(X) + \aleph_0 .
\end{aligned}$$

Hence

$$\mathrm{card}(N) \leqslant \sum_{n<\omega} \mathrm{card}(N_n) \leqslant \sum_{n<\omega} (\mathrm{card}(X) + \aleph_0) = \aleph_0 \cdot (\mathrm{card}(X) + \aleph_0) = \mathrm{card}(X) + \aleph_0 .$$

We prove the N-M-absoluteness of the $\in$-formula φ by induction on the complexity of φ. The cases $\varphi \equiv v_0 = v_1$ and $\varphi \equiv v_0 \in v_1$ are trivial. The induction steps for $\varphi \equiv \varphi_0 \vee \varphi_1$ and $\varphi \equiv \neg\varphi_0$ are easy. Finally consider the formula $\varphi \equiv \exists v_0\, \psi(v_0, v_1, \ldots, v_{k-1})$. Consider $a_1, \ldots, a_{k-1} \in N$. The Skolem value $u = S(\ulcorner\psi\urcorner, a_1, \ldots, a_{k-1})$ is an element of N. Then

$$\begin{aligned}
(\exists v_0\, \psi(v_0, a_1, \ldots, a_{k-1}))^N &\rightarrow \exists v_0 \in N\, \psi^N(v_0, a_1, \ldots, a_{k-1}) \\
&\rightarrow \exists v_0 \in N \psi^M(v_0, a_1, \ldots, a_{k-1}), \text{ by the inductive assumption,} \\
&\rightarrow \exists v_0 \in M \psi^M(v_0, a_1, \ldots, a_{k-1}) \\
&\rightarrow (\exists v_0\, \psi(v_0, a_1, \ldots, a_{k-1}))^M.
\end{aligned}$$

Conversely assume that $(\exists v_0\, \psi(v_0, a_1, \ldots, a_{k-1}))^M$. Then $I(M, \ulcorner\psi\urcorner, a_1, \ldots, a_{k-1}) \neq \emptyset$ and $z = S(\ulcorner\psi\urcorner, a_1, \ldots, a_{k-1})$ is the $\prec$-smallest element of M such that $\psi^M(z, a_1, \ldots, a_{k-1})$. The construction of N implies that $z \in N$. By induction hypothesis, $\psi^N(z, a_1, \ldots, a_{k-1})$. Hence $\exists v_0 \in N\, \psi^N(z, a_1, \ldots, a_{k-1}) \equiv (\exists v_0\, \psi(v_0, a_1, \ldots, a_{k-1}))^N$. $\square$

Note that this proof has some similarities with the proof of the Levy reflection principle. Putting $X = \emptyset$ the theorem implies that every formula that has some infinite model M has a *countable* model N. E.g., the formula "there is an uncountable set" has a countable model. This is the famous Skolem paradox. As a prepartation for the forcing method we also want the countable structure to be transitive.

Theorem 36. *Assume* $(Extensionality)^N$. *Then there is a transitive* $\bar{N}$ *and* $\pi: N \leftrightarrow \bar{N}$ *such that* π *is an* $\in$-isomorphism, *i.e.* $\forall x, y \in N\, (x \in y \leftrightarrow \pi(x) \in \pi(y))$. *Moreover,* $\bar{N}$ *and* π *are uniquely determined by* N. π *and* $\bar{N}$ *are called the* Mostowski transitivisation *or* collapse *of* N.

Proof. Define $\pi: N \to V$ recursively by

$$\pi(y) = \{\pi(x) | x \in y \cap N\}.$$

Set $\bar{N} = \pi[N]$.

(1) $\bar{N}$ is transitive.
Proof. Let $z \in \pi(y) \in \bar{N}$. Take $x \in y \cap N$ such that $z = \pi(x)$. Then $z \in \pi[N] = \bar{N}$. *qed*(1)

(2) $\pi: N \leftrightarrow \bar{N}$.
Proof. It suffices to show injectivity. Assume for a contradiction that $z \in \bar{N}$ is $\in$-minimal such that there are y, $y' \in N$, $y \neq y'$ with $z = \pi(y) = \pi(y')$. $(\text{Extensionality})^N$ implies $(\exists x (x \in y \leftrightarrow x \notin y'))^N$. Take $x \in N$ such that $x \in y \leftrightarrow x \notin y'$. We may assume that $x \in y$ and $x \notin y'$. Then $\pi(x) \in \pi(y) = \pi(y')$. According to the definition of π take $x' \in y' \cap N$ such that $\pi(x) = \pi(x')$. By the minimality of z, $x = x'$. But then $x = x' \in y'$, contradiction. *qed*(2)

(3) π is an $\in$-isomorphism.
Proof. Let $x, y \in N$. If $x \in y$ then $\pi(x) \in \pi(y)$ by the definition of π. Conversely assume that $\pi(x) \in \pi(y)$. By the definition of π take $x' \in y \cap N$ such that $\pi(x) = \pi(x')$. By (2), $x = x'$ and so $x \in y$. *qed*(3)

To show uniqueness assume that $\tilde{N}$ is transitive and $\tilde{\pi}: N \leftrightarrow \tilde{N}$ is an $\in$-isomorphism. Assume that $y \in N$ is $\in$-minimal such that $\pi(y) \neq \tilde{\pi}(y)$. We get a contradiction by showing that $\pi(y) = \tilde{\pi}(y)$. Consider $z \in \tilde{\pi}(y)$. The transitivity of $\tilde{N}$ implies $z \in \tilde{N}$. By the surjectivity of $\tilde{\pi}$ take $x \in N$ such that $z = \tilde{\pi}(x)$. Since $\tilde{\pi}$ is an $\in$-isomorphism, $x \in y$. And since π is an $\in$-isomorphism, $\pi(x) \in \pi(y)$. By the minimality of y, $\pi(x) = \tilde{\pi}(x)$. Hence $z = \tilde{\pi}(x) = \pi(x) \in \pi(y)$. Thus $\tilde{\pi}(y) \subseteq \pi(y)$. The converse can be shown analogously. Thus $\pi(y) = \tilde{\pi}(y)$, contradiction. □

It is easy to see that $\in$-isomorphisms preserve the truth of $\in$-formulas.

Lemma 37. *Let* $\pi: N \leftrightarrow \bar{N}$ *be an* $\in$*-isomorphism. Let* $\varphi(v_0, \ldots, v_{n-1})$ *be an* $\in$*-formula. Then*

$$\forall v_0, \ldots, v_{k-1} \in N\, (\varphi^N(v_0, \ldots, v_{k-1}) \leftrightarrow \varphi^{\bar{N}}(\pi(v_0), \ldots, \pi(v_{k-1}))).$$

Lemma 38. (ZFC) *Let* $\varphi_0, \ldots, \varphi_{n-1}$ *be* $\in$*-formulas without free variables with are true in* V*. Then there is a countable transitive set* $\bar{N}$ *such that* $\varphi_0^{\bar{N}}, \ldots, \varphi_{n-1}^{\bar{N}}$.

Proof. We may assume that φ_0 is the extensionality axiom. By the Reflection Theorem 7 we can take $\theta \in \mathrm{Ord}$ such that $\varphi_0^{V_\theta}, \ldots, \varphi_{n-1}^{V_\theta}$. By Theorem 35 there is a countable N such that all $\in$-formulas as N-V_θ-absolute. In particular $\varphi_0^N, \ldots, \varphi_{n-1}^N$. By Theorem 36 there is transitive set $\bar{N}$ and an $\in$-isomorphism $\pi\colon N \leftrightarrow \bar{N}$. Then $\bar{N}$ is countable. By Lemma 37 $\varphi_0^{\vec{N}}, \ldots, \varphi_{n-1}^{\bar{N}}$. $\square$

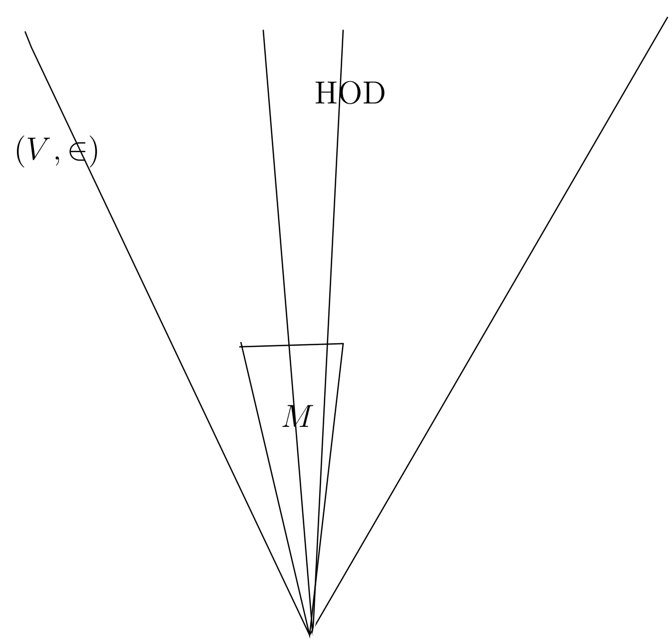
Theorem 39. *If* ZFC *is consistent then the following theory is also consistent:* $\mathrm{ZFC} + M$ *is countable and transitive* $+ \mathrm{ZFC}^M$*, where M is some variable.*

Proof. Assume that the theory $\mathrm{ZFC} + M$ is countable and transitive $+ \mathrm{ZFC}^M$ is inconsistent. Then there is a finite sequence $\varphi_0, \ldots, \varphi_{n-1}$ of ZFC-axioms such that the theory

$$\varphi_0, \ldots, \varphi_{n-1}, M \text{ is countable and transitive}, \varphi_0^M, \ldots, \varphi_{n-1}^M$$

implies $x \neq x$. Work in ZFC. By Lemma 38 there is a countable transitive set $\bar{N}$ such that $\varphi_0^{\vec{N}}, \ldots, \varphi_{n-1}^{\bar{N}}$. Setting $M = \bar{N}$ we get the contradiction $x \neq x$. Hence ZFC is inconsistent. $\square$

The consideration so far justify the following picture as a basis for further studies:



The argument of the Theorem can be extended to every $\in$-theory which extends ZFC, like $\mathrm{ZFC} + \mathrm{CH}$ or $\mathrm{ZFC} + \neg\mathrm{CH}$.

Theorem 40. *Let T be a theory in the language of set theory which extends* ZFC*. Assume that T is consistent. Then the following theory is also consistent: $T + M$ is countable and transitive $+ T^M$, where M is some variable.*

8 Extensions of Models of Set Theory

So far we have constructed and studied *inner models*, i.e. submodels of given models of set theory. We shall now work towards *extending* models of set theory by the *forcing* method of Paul Cohen. Cohen introduced these techniques to show the independence of AC and CH from ZF.

We shall work in the situation justified by Theorem 39: assume ZFC and ZFC^M where M is countable and transitive. Such an $\in$-structure $(M, \in)$ is called a *ground model*. We intend to adjoin a *generic set* G to M so that the extension $M[G]$ is again a model of ZFC. Cohen proved the independence of CH by constructing a *generic extension*

$$M[G] \vDash \mathrm{ZFC} + \neg \mathrm{CH}.$$

As already said in the introduction the extension $M \subseteq M[G]$ has some similarities to a transcendental field extension $k \subseteq k(a)$. The transcendental element a can be described in the ground field k by a variable x; some properties of a can be described in k. That $k(a)$ is a field follows from the field axioms in k. The extension is generated by k and a: every intermediate field K with $k \subseteq K \subseteq k(a)$ and $a \in K$ satisfies $K = k(a)$.

The settheoretic situation will be much more complicated than the algebraic analogue. Whereas there is up to isomorphism only one transcendental field extension of transcendence degree 1 we shall encounter a rich spectrum of generic extensions.

So fix the ground model M as above. We shall use sets G to determine extensions $M[G]$. G may be seen as the limit of a (countable) procedure in which more and more properties of $M[G]$ are being determinea or *forced*. Limits are often described by filters. Our G will be a filter on a preordering $(P, \leqslant)$.

Definition 41. *A* partial order *or a* forcing *is a tripel* $(P, \leqslant, 1_P)$ *such that* $(P, \leqslant)$ *is a transitive and reflexive binary relation (a* preordering*) with a maximal element* 1_P*. The elements of* P *are called* (forcing) conditions*. We say that* p *is* stronger than q *iff* $p \leqslant q$*. Conditions* $q_0, \ldots, q_{n-1}$ *are* compatible *iff they have a common* extension $p \leqslant q_0, \ldots, q_{n-1}$.

An example of a forcing relation is Cohen forcing $(P, \leqslant, 1_P)$:

$$P = \mathrm{Fn}(\omega, 2, \aleph_0) = \{p | p \colon \mathrm{dom}(p) \to 2 \wedge \mathrm{dom}(p) \subseteq \omega \wedge \mathrm{card}(\mathrm{dom}(p)) < \aleph_0\}$$

consists of all *partial* functions from ω to 2. Cohen forcing will approximate a total function from ω to 2, i.e. a *real* number. The approximation of a total function is captured by the forcing relation: a condition p is stronger than q iff the function p extends the function q:

$$p \leqslant q \text{ iff } p \supseteq q.$$

Let $1_P = \emptyset$ be the function with the least information content. Two COHEN condition q_1, q_2 are compatible iff they are compatible as functions, i.e. if $q_1 \cup q_2$ is a function.

Fix some forcing relation $(P, \leqslant, 1_P) \in M$. It is important that the forcing relation is an element of the ground model so that the ZFC-properties of M may be applied to P.

Definition 42. *$G \subseteq P$ is a* filter *on P iff*

a) $1_P \in G$;

b) $\forall q \in G \forall p \geqslant q\, p \in G$;

c) $\forall p, q \in G \exists r \in G\, (r \leqslant p \wedge r \leqslant q)$.

In the case of COHEN forcing, a filter is a system of pairwise compatible partial function whose union is again a partial function from ω to 2. We shall later introduce *generic* filters which would make that union a total function.

Fix a filter G on P. We shall construct an extension $M[G]$ which will satisfy some axioms of ZFC. This will later be strengthened to *generic extensions* which satisfy all of ZFC. Elements $x \in M[G]$ will have *names* $\dot{x} \in M$ in the ground model; G allows to *interpret* $\dot{x}$ as $x : x = \dot{x}^G$. The crucial issue for computing the interpretation $\dot{x}^G$ is to decide when $\dot{y}^G \in \dot{x}^G$. This shall be decided by the filter G. So the important information about $\dot{x}$ is contained in the set

$$\{(\dot{y}, p) | p \text{ decides that } \dot{y} \in \dot{x}\}.$$

In the forcing method one identifies $\dot{x}$ with that set:

$$\dot{x} = \{(\dot{y}, p) | p \text{ decides that } \dot{y} \in \dot{x}\}.$$

This motivates the following interpretation function:

Definition 43. *Define the G-*interpretation *$\dot{x}^G$ of $\dot{x} \in M$ by recursion on the strongly well-founded relation $\dot{y}\, R\, \dot{x}$ iff $\exists u\, (\dot{y}, u) \in \dot{x}$:*

$$\dot{x}^G = \{\dot{y}^G | \exists p \in G\ (\dot{y}, p) \in \dot{x}\}.$$

Let

$$M[G] = \{\dot{x}^G | \dot{x} \in M\}$$

be the extension *of M by P and G.*

We examine which settheoretic axioms hold in $M[G]$.

Lemma 44. *$M[G]$ is transitive.*

Proof. Let $u\in\dot{x}^G\in M[G]$. Then $u\in\{\dot{y}^G|\exists p\in G\ (\dot{y},p)\in\dot{x}\}\subseteq M[G]$. $\square$

Lemma 45. $\forall\dot{x}\in M\ \mathrm{rank}(\dot{x}^G)\leqslant\mathrm{rank}(\dot{x})$.

Proof. By induction on the relation $\dot{y}\,R\,\dot{x}$ iff $\exists u\ (\dot{y},u)\in\dot{x}$:

$$\begin{aligned}
\mathrm{rank}(\dot{x}^G) &= \bigcup\{\mathrm{rank}(\dot{y}^G)+1\,|\,\exists p\in G\ (\dot{y},p)\in\dot{x}\}\\
&\leqslant \bigcup\{\mathrm{rank}(\dot{y})+1\,|\,\exists p\in G\ (\dot{y},p)\in\dot{x}\}, \text{ by inductive hypothesis,}\\
&\leqslant \bigcup\{\mathrm{rank}((\dot{y},p))+1\,|\,(\dot{y},p)\in\dot{x}\}\\
&\leqslant \bigcup\{\mathrm{rank}(u)+1\,|\,u\in\dot{x}\}\\
&= \mathrm{rank}(\dot{x}).
\end{aligned}$$

$\square$

To show that $M[G]\supseteq M$ we define names for elements of M.

Definition 46. *Define by $\in$-recursion the* canonical name *for $x\in M$:*

$$\check{x}=\{(\check{y},1_P)\,|\,y\in x\}.$$

Lemma 47. *For $x\in M$ holds $\check{x}^G=x$. Hence $M\subseteq M[G]$.*

Proof. By $\in$-induction.

$$\begin{aligned}
\check{x}^G &= \{\dot{y}^G|\exists p\in G\ (\dot{y},p)\in\dot{x}\}\\
&= \{\check{y}^G|y\in x\}, \text{ by the definition of } \check{x} \text{ and since } 1_P\in G,\\
&= \{y|y\in x\}, \text{ by inductive hypothesis,}\\
&= x.
\end{aligned}$$

$\square$

Lemma 48. $M[G]\cap\mathrm{Ord}=M\cap\mathrm{Ord}$.

Proof. Let $\alpha\in M[G]\cap\mathrm{Ord}$. Take $\dot{x}\in M$ such that $\dot{x}^G=\alpha$. By Lemma 32, $\mathrm{rank}(u)$ is a definite term. Hence $\mathrm{rank}(\dot{x})\in M\cap\mathrm{Ord}$. Hence

$$\alpha=\mathrm{rank}(\alpha)=\mathrm{rank}(\dot{x}^G)\leqslant\mathrm{rank}(\dot{x})\in M\cap\mathrm{Ord}\,.$$ $\square$

To check that $G\in M[G]$ we need a name for G.

Definition 49. $\dot{G} = \{(\check{p}, p) | p \in P\}$ *is the* canonical name *for the filter on* P*.*

Lemma 50. $\dot{G} \in M$ *and* $\dot{G}^H = H$ *for any filter* H *on* P*.*

Proof. The term $\check{x}$ in the variable x is definite since it is defined by a definite $\in$-recursion. So $(\check{x}, x)$ and $\{(\check{p}, p) | p \in P\}$ are definite terms in the variables x and P resp. Then $P \in M$ implies that $\dot{G} \in M$. Moreover

$$\dot{G}^H = \{\check{p}^H | p \in H\} = \{p | p \in H\} = H.$$

$\square$

Theorem 51. $M[G]$ *is a model of Extensionality, Pairing, Union, Infinity, and Foundation.*

Proof. We employ the criteria of Theorem 2. Extensionality and Choice hold since $M[G]$ is a transitive $\in$-model.

Pairing: Let $x, y \in M[G]$. Take names $\dot{x}, \dot{y} \in M$ such that $x = \dot{x}^G$, $y = \dot{y}^G$. Set

$$\dot{z} = \{(\dot{x}, 1_P), (\dot{y}, 1_P)\}.$$

Then

$$\{x, y\} = \{\dot{x}^G, \dot{y}^G\} = \dot{z}^G \in M[G].$$

Union: Let $x \in M[G]$ and $x = \dot{x}^G$, $\dot{x} \in M$. Set

$$\dot{z} = \{(\dot{u}, r) | \exists p, q \in P \exists \dot{v} (r \leqslant p \wedge r \leqslant q \wedge (\dot{u}, p) \in \dot{v} \wedge (\dot{v}, q) \in \dot{x}\}.$$

The right-hand side is a definite term in the variables $P, \leqslant, \dot{x} \in M$, hence $\dot{z} \in M$. We show that $\bigcup x = \dot{z}^G$.

Let $u \in \bigcup x$. Take $v \in x$ such that $u \in v \in x = \dot{x}^G$. Take $\dot{v} \in M$ and $q \in G$ such that $(\dot{v}, q) \in \dot{x}$ and $\dot{v}^G = v$. Take $\dot{u} \in M$ and $p \in G$ such that $(\dot{u}, p) \in \dot{v}$ and $\dot{u}^G = u$. Take $r \in G$ such that $r \leqslant p, q$. By the definition of $\dot{z}$, $(\dot{u}, r) \in \dot{z}$, and $u = \dot{u}^G \in \dot{z}^G$ since $r \in G$.

Conversely let $u \in \dot{z}^G$. Take $r \in G$ and $\dot{u} \in M$ such that $(\dot{u}, r) \in \dot{z}$ and $u = \dot{u}^G$. By the definition of $\dot{z}$, take $p, q \in P$ and $\dot{v} \in M$ such that

$$r \leqslant p \wedge r \leqslant q \wedge (\dot{u}, p) \in \dot{v} \wedge (\dot{v}, q) \in \dot{x}.$$

Then $p, q \in G$ and $u = \dot{u}^G \in \dot{v}^G \in \dot{x}^G = x$. Hence $u \in \bigcup x$.

Infinity holds in $M[G]$ since $\omega \in M \subseteq M[G]$. $\square$

Problem 1. Do Powerset and Choice hold in $M[G]$?

9 Generic Filters and the Forcing Relation

If $(\dot{y}, p) \in \dot{x}$ then $p \in H \to \dot{y}^H \in \dot{x}^H$; so regardless of other aspects p "forces" that $\dot{y} \in \dot{x}$. And if $\dot{y}^H \in \dot{x}^H$ this is (leaving some technical issues aside) forced by some $p \in H$. We want to generalise this phenomenon from the most fundamental of furmulas, $v_0 \in v_1$, to all $\in$-formulas: consider a formula $\varphi(v_0, \ldots, v_{n-1})$ and names $\dot{x}_0, \ldots, \dot{x}_{n-1}$. We want a relation

$$p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$$

such that

a) $p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$ implies that $M[H] \vDash \varphi(\dot{x}_0^H, \ldots, \dot{x}_{n-1}^H)$ for every appropriate filter H on P with $p \in H$

b) if $M[H] \vDash \varphi(\dot{x}_0^H, \ldots, \dot{x}_{n-1}^H)$ for some appropriate filter H on P with $p \in H$ then there is $p \in H$ such that $p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$.

Let us continue the discussion with the vague notion of "appropriate filter". By b), an appropriate filter H has to decide every φ. There is $r \in H$ such that $r \Vdash \varphi$ or $r \Vdash \neg\varphi$:

$$\{r \in P | r \Vdash \varphi \text{ or } r \Vdash \neg\varphi\} \cap H \neq \emptyset;$$

We argue that the set $D = \{r \in P | r \Vdash \varphi \text{ or } r \Vdash \neg\varphi\}$ is a *dense* set in P. Let $p \in P$. Take an appropriate filter H on P with $p \in H$. Suppose that $M[H] \vDash \varphi$. By b) take some $q \in H$ such that $q \Vdash \varphi$. By the compatibility of filter elements take $r \in H$ such that $r \leqslant p, q$. Then $r \Vdash \varphi$ and $r \in D$. In case $M[H] \vDash \neg\varphi$ we similarly find $r \leqslant p$, $r \in D$.

It will turn out that the set D will be definable inside the ground model, thus $D \in M$. Accordingly, a filter H on P will be appropriate if it intersects *every* $D \in M$ which is a dense subset of P. We now give rigorous definitions of appropriate filters and of the forcing relation.

Definition 52. *Let* $(P, \leqslant, 1_P)$ *be a forcing.*

a) $D \subseteq P$ *is* dense *in* P *iff* $\forall p \in P \exists q \in D \, q \leqslant p$.

b) A filter G *on* P *is* M-generic *iff* $D \cap G \neq \emptyset$ *for every* $D \in M$ *which is dense in* P.

If $M[G]$ *is an extension of* M *by an* M*-generic filter we call* $M[G]$ *a* generic extension.

For *countable* ground models we have

Theorem 53. *Let* $(P,\leqslant,1_P)$ *be a partial order, let* M *be a countable ground model, and let* $p\in P$*. Then there is an* M*-generic filter* G *on* P *with* $p\in G$*.*

Proof. Take a wellorder $\prec$ of M in ordertype ω. Let $(D_n|n<\omega)$ be an enumeration of all $D\in M$ which are dense in P. Define an ω-sequence $p=p_0\geqslant p_1\geqslant p_2\geqslant\ldots$ recursively:

$$p_{n+1} \text{ is the } \prec\text{-smallest element of } M \text{ such that } p_{n+1}\leqslant p_n \text{ and } p_{n+1}\in D_n.$$

Then $G=\{p\in P|\exists n<\omega\, p_n\leqslant p\}$ is as desired. $\square$

Fix a ground model M and a partial order $(P,\leqslant,1_P)\in M$.

Definition 54. *Let* $\varphi(\dot{x}_0,\ldots,\dot{x}_{n-1})$ *be a* sentence of the forcing language*, i.e.* $\varphi(v_0,\ldots,v_{n-1})$ *is an* $\in$*-formulas and* $\dot{x}_0,\ldots,\dot{x}_{n-1}\in M$*. For* $p\in P$ *define* $p\Vdash_P^M\varphi(\dot{x}_0,\ldots,\dot{x}_{n-1})$*,* p forces $\varphi(\dot{x}_0,\ldots,\dot{x}_{n-1})$*, iff for all* M*-generic filters* G *on* M *with* $p\in G$*:*

$$\varphi^{M[G]}(\dot{x}_0^G,\ldots,\dot{x}_{n-1}^G).$$

If M *or* P *are obvious from the context we also write* $\Vdash_P$ *or* $\Vdash$ *instead of* $\Vdash_P^M$*.*

We shall state several properties of $\Vdash$. Some of the properties amount to a definition of $\Vdash\varphi$ by recursion on the complexity of φ which can be carried out inside the ground model M.

Lemma 55.

a) *If* $p\Vdash\varphi$ *and* $q\leqslant p$ *then* $q\Vdash\varphi$*.*

b) *If* $p\Vdash\varphi$ *and* φ *implies* ψ *then* $p\Vdash\psi$*.*

c) *If* $(\dot{y},p)\in\dot{x}$ *and* $p\in P$ *then* $p\Vdash\dot{y}\in\dot{x}$*.*

Proof. *a)* Let $G\ni q$ be M-generic on P. Then $p\in G$. Hence $M[G]\vDash\varphi$.

b) Let $G\ni p$ be M-generic on P. Then $M[G]\vDash\varphi$. Since φ implies ψ, also $M[G]\vDash\psi$.

c) Let $G\ni p$ be M-generic on P. Then

$$\dot{y}^G\in\{\dot{u}^G|\exists q\in G\ (\dot{u},q)\in\dot{x}\}=\dot{x}^G.$$ $\square$

For simplicity we assume that $\in$-formulas are only built from the connectives $\wedge,\neg,\forall$. We want to show (recursively) that every $\in$-formula has the following property:

Definition 56. *The* $\in$*-formula* $\varphi(v_0, \ldots, v_{n-1})$ satisfies the forcing theorem *iff the following hold:*

a) *The class*

$$\mathrm{Force}_{\varphi} = \{(p, \dot{x}_0, \ldots, \dot{x}_{n-1}) \,|\, p \in P \wedge \dot{x}_0, \ldots, \dot{x}_{n-1} \in M \wedge p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})\}$$

is definable in M;

b) *if* $M[G]$ *is a generic extension and* $\dot{x}_0, \ldots, \dot{x}_{n-1} \in M$ *with* $M[G] \vDash \varphi(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$ *then there is* $p \in G$ *such that* $p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$.

Lemma 57. *Let* $\varphi(v_0, \ldots, v_{n-1})$ *and* $\psi(v_0, \ldots, v_{n-1})$ *be* $\in$*-formulas satisfying the forcing theorem. Then we have for all names* $\dot{x}_0, \ldots, \dot{x}_{n-1} \in M$

a) $p \Vdash (\varphi \wedge \psi)(\dot{x}_0, \ldots, \dot{x}_{n-1})$ *iff* $p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$ *and* $p \Vdash \psi(\dot{x}_0, \ldots, \dot{x}_{n-1})$.

b) $p \Vdash \neg\varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$ *iff* $\forall q \leqslant p \neg q \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$.

c) $p \Vdash \forall v_0 \varphi(v_0, \dot{x}_1, \ldots, \dot{x}_{n-1})$ *iff* $\forall \dot{x}_0 \in M\, p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$.

d) *The formulas* $(\varphi \wedge \psi)$, $\neg\varphi$, *and* $\forall v_0\, \varphi$ *satisfy the forcing theorem.*

Proof. *a*) is immediate.

b) For the implication from left to right assume $p \Vdash \neg\varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$ and let $q \leqslant p$. If $q \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$ then $p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$. Take an M-generic $G \ni p$. Then $M[G] \vDash \neg \varphi(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$ and $M[G] \vDash \varphi(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$. Contradiction.

For the converse assume $\neg p \Vdash \neg\varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$. By the definition of $\Vdash$ take an M-generic $G \ni p$ such that $M[G] \vDash \varphi(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$. Since φ satisfies the forcing theorem take $r \in G$ with $r \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$. Take $q \in G$ such that $q \leqslant p, r$. Then $q \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$, and the right-hand side of the equivalence is false.

c) is similar to the case *a*). The implication from left to right is immediate. For the converse assume $\forall \dot{x}_0 \in M\, p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$. et $G \ni p$ be M-generic on P. Then $\forall \dot{x}_0 \in M\, M[G] \vDash \varphi(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$. Then $M[G] \vDash \forall v_0 \varphi(v_0, \dot{x}_1^G, \ldots, \dot{x}_{n-1}^G)$. Thus $p \Vdash \forall v_0 \varphi(v_0, \dot{x}_1, \ldots, \dot{x}_{n-1})$.

d) The cases *a*) $-$ *c*) contain definitions of $\mathrm{Force}_{\varphi \wedge \psi}$, $\mathrm{Force}_{\neg\varphi}$, and $\mathrm{Force}_{\forall v_0 \varphi}$ on the basis of definitions of Force_{φ} and Force_{ψ}. We now show *b*) of Definition 56 for $\varphi \wedge \psi$, $\neg\varphi$, and $\forall v_0\, \varphi$. So let $M[G]$ be a generic extension.

$\varphi \wedge \psi$: Assume $M[G] \vDash (\varphi \wedge \psi)(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$. Then $M[G] \vDash \varphi(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$ and $M[G] \vDash \psi(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$. Since φ and ψ satisfy the forcing theorem, take $p, q \in G$ such that $p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$ and $q \Vdash \psi(\dot{x}_0, \ldots, \dot{x}_{n-1})$. Take $r \in G$ with $r \leqslant p, q$. Then $r \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$, $r \Vdash \psi(\dot{x}_0, \ldots, \dot{x}_{n-1})$, and $r \Vdash (\varphi \wedge \psi)(\dot{x}_0, \ldots, \dot{x}_{n-1})$.

$\neg\varphi$: Assume $M[G]\vDash\neg\varphi(\dot{x}_0^G,\ldots,\dot{x}_{n-1}^G)$. Define

$$D=\{p\in P\,|\,p\Vdash\varphi(\dot{x}_0,\ldots,\dot{x}_{n-1})\text{ or }\forall q\leqslant p\ \ \neg q\Vdash\varphi(\dot{x}_0,\ldots,\dot{x}_{n-1})\}.$$

Since Force_φ is definable in M, we get $D\in M$. It is easy to see that D is dense in P. By the genericity of G take $p\in G\cap D$. We cannot have $p\Vdash\varphi(\dot{x}_0,\ldots,\dot{x}_{n-1})$ because $M[G]\vDash\neg\varphi(\dot{x}_0^G,\ldots,\dot{x}_{n-1}^G)$. Hence $\forall q\leqslant p\ \ \neg q\Vdash\varphi(\dot{x}_0,\ldots,\dot{x}_{n-1})$. Then *b*) implies that $p\Vdash\neg\varphi(\dot{x}_0,\ldots,\dot{x}_{n-1})$.

$\forall v_0\,\varphi$: Assume $M[G]\vDash\forall v_0\,\varphi(v_0,\dot{x}_1^G,\ldots,\dot{x}_{n-1}^G)$. Define

$$D=\{p\in P\,|\forall\dot{x}_0\in M\,p\Vdash\varphi(\dot{x}_0,\ldots,\dot{x}_{n-1})\text{ or }\exists\dot{x}_0\in M\,p\Vdash\neg\varphi(\dot{x}_0,\dot{x}_1\ldots,\dot{x}_{n-1})\}.$$

Then $D\in M$ since Force_φ and $\mathrm{Force}_{\neg\varphi}$ are definable in M.

(1) D is dense in P.
Proof. Consider $r\in P$. If $\forall\dot{x}_0\in M\,r\Vdash\varphi(\dot{x}_0,\ldots,\dot{x}_{n-1})$ then $r\in D$. Otherwise take $\dot{x}_0\in M$ with $\neg r\Vdash\varphi(\dot{x}_0,\ldots,\dot{x}_{n-1})$. Take an M-generic filter $H\ni r$ such that $M[H]\vDash\neg\varphi(\dot{x}_0^G,\ldots,\dot{x}_{n-1}^G)$. Since $\neg\varphi$ satisfies the forcing theorem, take $s\in H$ with $s\Vdash\neg\varphi(\dot{x}_0,\ldots,\dot{x}_{n-1})$. Take $p\in H$ such that $p\leqslant r,s$. Then $p\Vdash\neg\varphi(\dot{x}_0,\ldots,\dot{x}_{n-1})$ and $p\in D$. *qed*(1)

By the genericity of G take $p\in G\cap D$. Assume for a contradiction that $\exists\dot{x}_0\in M\,p\Vdash\neg\varphi(\dot{x}_0,\dot{x}_1\ldots,\dot{x}_{n-1})$. Take $\dot{x}_0\in M$ such that $p\Vdash\neg\varphi(\dot{x}_0,\dot{x}_1\ldots,\dot{x}_{n-1})$. Since $p\in G$, $M[G]\vDash\neg\varphi(\dot{x}_0^G,\dot{x}_1^G,\ldots,\dot{x}_{n-1}^G)$, contradicting the assumption of the quantifier case. So p is in the "other half" of D, i.e. $\forall\dot{x}_0\in M\,p\Vdash\varphi(\dot{x}_0,\ldots,\dot{x}_{n-1})$. By *c*), $p\Vdash\forall v_0\,\varphi(v_0,\dot{x}_1,\ldots,\dot{x}_{n-1})$. $\square$

10 The Atomic Case

The atomic case of the forcing theorem turns out more complicated than the cases that we have considered so far. This is due to the hierarchical structure of sets. We treat the equality case $v_1=v_2$ as two inclusions $v_1\subseteq v_2$ and $v_2\subseteq v_1$. The relation $x_1^G\subseteq x_2^G$ is equivalent to

$$\{y_1^G|\exists s_1\in G\ (y_1,s_1)\in x_1\}\subseteq\{y_2^G|\exists s_2\in G\ (y_2,s_2)\in x_2\}.$$

Lemma 58.

a) $p\Vdash x_1\subseteq x_2$ *iff* $\forall(y_1,s_1)\in x_1\,(s_1\in P\rightarrow$
$D(y_1,s_1,x_2):=\{q\in P|q\leqslant s_1\rightarrow\exists(y_2,s_2)\in x_2(s_2\in P\wedge q\leqslant s_2\wedge q\Vdash y_1\subseteq y_2\wedge q\Vdash y_2\subseteq y_1)\}$
is dense in P below p).

b) $\mathrm{Force}_{v_1\subseteq v_2}$ *is definable in M.*

c) *If $x_1^G\subseteq x_2^G$ then there is $p\in G$ such that $p\Vdash x_1\subseteq x_2$.*

Here we say that a set $D \subseteq P$ is dense in P below p *iff* $\forall p' \leqslant p \exists q \leqslant q\, q \in D$.

Proof. Consider the relation

$$(q, y_1, y_2)\, R\, (p, x_1, x_2) \leftrightarrow (y_1 \in \mathrm{dom}(x_1) \vee y_1 \in \mathrm{dom}(x_2)) \wedge (y_2 \in \mathrm{dom}(x_1) \vee y_2 \in \mathrm{dom}(x_2)).$$

(1) R is strongly wellfounded.
Proof. If $(q, y_1, y_2)\, R\, (p, x_1, x_2)$ then

$$(\mathrm{rg}(y_1) < \mathrm{rg}(x_1) \vee \mathrm{rg}(y_1) < \mathrm{rg}(x_2)) \wedge (\mathrm{rg}(y_2) < \mathrm{rg}(x_1) \vee \mathrm{rg}(y_2) < \mathrm{rg}(x_2)),$$

and so $\max(\mathrm{rg}(y_1), \mathrm{rg}(y_2)) < \max(\mathrm{rg}(x_1), \mathrm{rg}(x_2))$. Hence an infinite decreasing sequence in R leads to an infinite decreasing sequence in Ord. *qed*(1)

By recursion on R define

$$\begin{aligned} S(p, x_1, x_2) \;\leftrightarrow\; & \forall (y_1, s_1) \in x_1\, (s_1 \in P \rightarrow \\ & \{q \in P \mid q \leqslant s_1 \rightarrow \exists (y_2, s_2) \in x_2 (s_2 \in P \wedge q \leqslant s_2 \wedge S(q, y_1, y_2) \wedge S(q, y_2, y_1)\} \\ & \text{is dense in } P \text{ below } p). \end{aligned}$$

By a simultaneous induction on R we prove that $(p \Vdash x_1 \subseteq x_2) \leftrightarrow S(p, x_1, x_2)$ and properties *a*) and *c*). This also proves *b*).

a) Assume $p \Vdash x_1 \subseteq x_2$. Let $(y_1, s_1) \in x_1$ and $s_1 \in P$. To show that $D(y_1, s_1, x_2)$ is dense in P below p consider $p' \leqslant p$. It suffices to find $q \leqslant p'$ with $q \in D(y_1, s_1, x_2)$. Let $G \ni p'$ be M-generic on P.

If $\neg p' \leqslant s_1$ then $p' \in D(y_1, s_1, x_2)$ and we can take $q = p'$.

So assume that $p' \leqslant s_1$. Then $s_1, p \in G$ and

$$y_1^G \in x_1^G \subseteq x_2^G = \{y_2^G \mid \exists s_2 \in G\ (y_2, s_2) \in x_2\}.$$

Take $(y_2, s_2) \in x_2$ such that $s_2 \in G$ and $y_1^G = y_2^G$. Then $y_1^G \subseteq y_2^G$ and $y_2^G \subseteq y_1^G$. By the inductive assumption c) take $p'', p''' \in G$ such that $p'' \Vdash y_1 \subseteq y_2$ and $p''' \Vdash y_2 \subseteq y_1$. Take $q \in G$ such that $q \leqslant p', s_2, p'', p'''$. Then $q \leqslant p' \leqslant s_1$, $q \leqslant s_2$, $q \Vdash y_1 \subseteq y_2$, and $q \Vdash y_2 \subseteq y_1$. Hence $q \in D(y_1, s_1, x_2)$.

Conversely assume the right-hand side of *a*). Let $G \ni p$ be M-generic on P. We have show that $x_1^G \subseteq x_2^G$, i.e. $\{y_1^G \mid \exists s_1 \in G\ (y_1, s_1) \in x_1\} \subseteq \{y_2^G \mid \exists s_2 \in G\ (y_2, s_2) \in x_2\}$. So let $y_1^G \in x_1^G$. Take $s_1 \in G$ such that $(y_1, s_1) \in x_1$. Take $p' \in G$, $p' \leqslant p, s_1$. The right-hand side of a) implies that $D(y_1, s_1, x_2)$ is dense in P below p and thus below p'. By the inductive assumption, $D(y_1, s_1, x_2) \in M$. By the genericity of G, take $q \in G$, $q \leqslant p'$, $q \in D(y_1, s_1, x_2)$. By the definition of $D(y_1, s_1, x_2)$ take $(y_2, s_2) \in x_2$ such that

$$s_2 \in P \wedge q \leqslant s_2 \wedge q \Vdash y_1 \subseteq y_2 \wedge q \Vdash y_2 \subseteq y_1.$$

Since $q, s_2 \in G$ this implies $y_1^G \subseteq y_2^G$, $y_2^G \subseteq y_1^G$, and so

$$y_1^G = y_2^G \in x_2^G.$$

Ror $(q,\ y_1,\ y_2)\ R\ (p,\ x_1,\ x_2)$ the induction hypothesis implies that $S(q,\ y_1,\ y_2)$ and $S(q,\ y_2, y_1)$ agree with $q \Vdash y_1 \subseteq y_2$ and $q \Vdash y_2 \subseteq y_1$ respectively. Now $a)$ and the recursive definition of $S(p, x_1, x_2)$ agree and yield that

$$(p \Vdash x_1 \subseteq x_2) \leftrightarrow S(p, x_1, x_2).$$

$c)$ Let $M[G]$ be a generic extension such that $M[G] \vDash x_1^G \subseteq x_2^G$. Set

$$\begin{aligned} D = \{p \in P \ \mid \ & p \Vdash x_1 \subseteq x_2 \\ & \vee \exists (y_1, s_1) \in x_1 \, (s_1 \in P \wedge \forall q \leqslant p \\ & \ (q \leqslant s_1 \wedge \forall (y_2, s_2) \in x_2 ((s_2 \in P \wedge q \Vdash y_1 \subseteq y_2 \wedge q \Vdash y_2 \subseteq y_1) \rightarrow \neg q \leqslant s_2)))\}. \end{aligned}$$

$D \in M$ since by the inductive assumption we may replace $\Vdash$ in the definition of D by the predicate S which is definable in M.

(2) D is dense in P.
Proof. Let $r \in P$. If $r \Vdash x_1 \subseteq x_2$ we are done. So assume $\neg r \Vdash x_1 \subseteq x_2$. By the equivalence in $a)$ take $(y_1, s_1) \in x_1$ such that $s_1 \in P$ and $D(y_1, s_1, x_2)$ is *not* dense in P below r. Take $p \leqslant r$ such that $\forall q \leqslant p \, q \notin D(y_1, s_1, y_2)$. $q \notin D(y_1, s_1, y_2)$ is equivalent to

$$q \leqslant s_1 \wedge \forall (y_2, s_2) \in x_2 (s_2 \in P \wedge q \Vdash y_1 \subseteq y_2 \wedge q \Vdash y_2 \subseteq y_1 \rightarrow \neg q \leqslant s_2).$$

Hence $p \leqslant r$ is an element of D. *qed*(2)

By the M-genericity take $p \in G \cap D$. We claim that $p \Vdash x_1 \subseteq x_2$. If not then the alternative in the definition of D holds: take $(y_1, s_1) \in x_1$ such that $s_1 \in P$ and

$$\forall q \leqslant p \ (q \leqslant s_1 \wedge \forall (y_2, s_2) \in x_2 ((s_2 \in P \wedge q \Vdash y_1 \subseteq y_2 \wedge q \Vdash y_2 \subseteq y_1) \rightarrow \neg q \leqslant s_2)). \tag{5}$$

In particular for $q = p$ we have

$$p \leqslant s_1 \wedge \forall (y_2, s_2) \in x_2 ((s_2 \in P \wedge p \Vdash y_1 \subseteq y_2 \wedge p \Vdash y_2 \subseteq y_1) \rightarrow \neg p \leqslant s_2).$$

Then $s_1 \in G$ and $y_1^G \in x_1^G \subseteq x_2^G = \{y_2^G | \exists s_2 \in G \ (y_2, s_2) \in x_2\}$. Take $(y_2, s_2) \in x_2$ such that $s_2 \in G$ and $y_1^G = y_2^G$. Then $y_1^G \subseteq y_2^G$ and $y_2^G \subseteq y_1^G$. Since $c)$ holds at R-smaller triples, there are $q', q'' \in G$ such that $q' \Vdash y_1 \subseteq y_2$ and $q'' \Vdash y_2 \subseteq y_1$. Take $q \in G$ such that $q \leqslant p, s_2, q', q''$. Then (y_2, s_2) satisfies

$$s_2 \in P \wedge q \Vdash y_1 \subseteq y_2 \wedge q \Vdash y_2 \subseteq y_1 \wedge q \leqslant s_2.$$

But this contradicts (5). Hence $p \Vdash x_1 \subseteq x_2$. $\square$

We can now deal with the other atomic cases:

Lemma 59.

a) $x = y$ *satisfies the forcing theorem.*

b) $x \in y$ *satisfies the forcing theorem.*

Proof. For *a*) observe that $p \Vdash x = y$ iff $p \Vdash x \subseteq y$ and $p \Vdash y \subseteq x$.

b) We claim that $p \Vdash x \in y$ iff $D = \{q \leqslant p | \exists (u, r) \in y\ (q \leqslant r \wedge q \Vdash x = u)\}$ is dense in P below p.

Assume that $p \Vdash x \in y$. To prove the density of D consider $s \leqslant p$. Take an M-generic filter G on P with $s \in G$. $s \Vdash x \in y$ and so $x^G \in y^G = \{u^G | \exists r \in G\ (u, r) \in y\}$. Take $(u, r) \in y$ such that $x^G = u^G$ and $r \in G$. By the forcing theorem for equalities take $t \in G$ such that $t \Vdash x = u$. Take $q \in G$ such that $q \leqslant s, r, t$. Then $q \leqslant p$, $q \leqslant r$, and $q \Vdash x = u$. Hence $q \in D$.

Conversely let D be dense in P below p. To show that $p \Vdash x \in y$ let G be an M-generic filter on P with $p \in G$. By the genericity there is $q \leqslant p$ such that $q \in G \cap D$. Take $(u, r) \in y$ such that $q \leqslant r \wedge q \Vdash x = u$. Then $r \in G$ and $x^G = u^G \in y^G$.

Finally assume that $x^G \in y^G$. $y^G = \{u^G | \exists r \in G\ (u, r) \in y\}$. Take some $(u, r) \in y$ such that $r \in G$ and $x^G = u^G$. By *a*) take $s \in G$ such that $s \Vdash x = u$. Take $p \in G$ such that $p \leqslant r, s$. Then $p \Vdash x = u$ and $p \Vdash u \in y$. Hence $p \Vdash x \in y$. $\square$

So we have proved the *forcing theorem*:

Theorem 60. *For every* $\in$*-formula* $\varphi(v_0, \ldots, v_{n-1})$ *the following hold:*

a) *The property* $p \Vdash_P^M \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$ *is definable in* M*;*

b) *if* $M[G] \vDash \varphi(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$ *in a generic extension* $M[G]$ *then there is* $p \in G$ *such that* $p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$.