Models of Set Theory II

by Peter Koepke

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Abstract

Martin's Axiom and some consequences, iterated forcing, forcing Martin's axiom, Cichon's diagram, proper forcing, supercompact cardinals, forcing the Proper Forcing Axiom

1 Introduction

The method of *forcing* allows to construct models of axioms of set theory with interesting or exotic properties. Certain results can be obtained by *transfinite iterations* of this technique. More precisely, iterated forcing defines generic extensions, which can be construed as an increasing well-ordered tower of intermediate models where successor models are generic extensions of the previous model. Such an analysis is already possible for the standard Cohen model for $2^{\aleph_0} = \aleph_2$. In that model, *partially* generic filters exist for the standard Cohen forcing $\mathrm{Fn}(\aleph_0, 2, \aleph_0)$. This motivates *forcing axioms* postulating the existence of partially generic filters for certain forcings. *Martin's Axiom* MA is a forcing axiom for forcings satisfying the countable antichain condition (ccc). We shall study some consequences of MA and shall then force MA by iterated forcing. We shall also study the *Proper Forcing Axiom* PFA for a class of forcings which are *proper*.

Many forcing constructions are concerned with properties of the set $\mathbb{R}$ of real numbers. There are several forcings which adjoin new reals to (ground) models. Different forcings adjoin reals which may be very different with respect to growth behaviour and other aspects. Cardinal characteristics of $\mathbb{R}$ have been introduced to describe such behaviours. They are systematised in Cichon's diagram. Using MA and iterated forcings several constellations of cardinals are realizable in Cichon's diagram.

2 Cohen forcing

The most basic forcing construction is the adjunction of a Cohen generic real c to a countable transitive ground model M. The generic extension $M[c]$ is again a countable transitive model of ZFC and it contains the "new" real $c \notin M$. We saw before that the adjunction of c has consequences for the set theory within $M[c]$:

Theorem 1. *In the* Cohen *extension $M[c]$ the set $\mathbb{R} \cap M$ of ground model reals has (Lebesgue) measure zero.*

On the other hand, inside a given model of set theory, the set of reals has positive measure, i.e., does not have measure measure.

Exercise 1. Show that the measure zero sets form a proper ideal on $\mathbb{R}$ which is closed under countable unions.

Exercise 2. Show that the following *Cantor set* of reals has cardinality $2^{\aleph_0}$ and measure zero:

$$C = \{x \in \mathbb{R} \mid \forall n < \omega x(2n) = x(2n+1)\}.$$

The set of ground model reals in $M[c]$ can be a set of size $\aleph_1$ that has measure zero. This leads to the question whether it is (relatively) consistent that *all* sets of reals of size $\aleph_1$ have measure zero. Of course this necessitates $2^{\aleph_0} > \aleph_1$. It is natural to ask the question about Cohen's canonical model for $2^{\aleph_0} > \aleph_1$.

Consider adjoining λ Cohen reals to a ground model M where $\lambda = \aleph_2^M$. Define λ-fold Cohen forcing $P = (P, \leqslant, 1) \in M$ by $P = \mathrm{Fn}(\lambda \times \omega, 2, \aleph_0)$, $\leqslant = \supseteq$, and $1 = \emptyset$. Let G be M-generic on P. Let $F = \bigcup G \colon \lambda \times \omega \to 2$ and extract a sequence $(c_\beta | \alpha < \lambda)$ of Cohen reals $c_\beta \colon \omega \to 2$ from F by:

$$c_\beta(n) = F(\beta, n).$$

Then the generic extension is generated by the sequence of Cohen reals:

$$M[G] = M[(c_\beta | \beta < \lambda)].$$

It is natural to construe $M[G]$ as a limit of the models $M[(c_\beta | \beta < \alpha)]$ when α goes towards λ: Fix $\alpha \leqslant \lambda$. Let $P_\alpha = \mathrm{Fn}(\alpha \times \omega, 2, \aleph_0)$ and $R_\alpha = \mathrm{Fn}((\lambda \setminus \alpha) \times \omega, 2, \aleph_0)$, partially ordered by reverse inclusion. The isomorphisms

$$P \cong P_\alpha \times R_\alpha \text{ and } P_{\alpha+1} \cong P_\alpha \times Q$$

imply that $G_\alpha = G \cap P_\alpha$ is M-generic on P_α and that

$$H_\alpha = \{q \in Q \mid \{((\alpha, n), i) \mid (n, i) \in q\} \in G_{\alpha+1}\}$$

is $M[G_\alpha]$-generic on Q. Let $M_\alpha = M[G_\alpha]$ be the α-th model in this construction. Then

$$M_{\alpha+1} = M[G_{\alpha+1}] = M[G_\alpha][H_\alpha] = M_\alpha[H_\alpha].$$

It is straightforward to check that $c_\alpha = \bigcup H_\alpha$. So the model $M[G] = M_\lambda$ is obtained by a sequence of models $(M_\alpha \mid \alpha \leqslant \lambda)$ where each successor step is a Cohen extension of the previous step. The whole construction is held together by the "long" generic set G which dictates the sequence of the construction and also the behaviour at limit stages.

Consider a real $x \in M[G]$. Identifying characteristic functions with sets we can view x as a subset of ω. In the previous course we had seen that there is a name $\dot{x} \in M$, $\dot{x}^G = x$ of the form

$$\dot{x} = \{(\check{n}, q) | n < \omega \wedge q \in A_n\},$$

where every A_n is an antichain in P. Since P satisfies the countable chain condition, there is $\alpha < \lambda$ such that $A_n \subseteq P_\alpha$ for every $n < \omega$. Then

$$x = \dot{x}^G = \dot{x}^{(G \cap P_\alpha)} = \dot{x}^{G_\alpha} \in M[G_\alpha]$$

In $M[G]$ consider a set $B = \{x_i \mid i < \aleph_1\}$ of reals of size $\aleph_1$. One can view B as a subset of $\aleph_1^M$. As in the above argument, there is an $\alpha < \lambda$ such that $B \in M_\alpha$. By our previous Lemma, $B \subseteq \mathbb{R} \cap M_\alpha$ has measure zero in the Cohen generic extension $M[c_\alpha]$. So B has measure zero in $M[G]$. The model $M[G]$ establishes:

Theorem 2. *If* ZFC *is consistent then* ZFC + *"every set of reals of size* $\leqslant \aleph_1$ *has Lebesgue measure zero" is consistent.*

Together with models of the Continuum Hypothesis this shows that the statement "every set of reals of size $\leqslant \aleph_1$ has Lebesgue measure zero" is independent of the axioms of ZFC.

One can ask for further properties of Lebesgue measure in connection with the uncountable. Is it consistent that every union of an $\aleph_1$-sequence of measure zero sets has again measure zero?

Exercise 3.

a) Show that in the model $M[G] = M[(c_\beta \mid \beta < \lambda)]$ there is an $\aleph_1$-sequence of measure zero sets whose union is $\mathbb{R}$.

b) Show that $\{c_\beta \mid \beta < \lambda\}$ has measure zero in $M[G]$.

Exercise 4. Define forcing with sets of reals of *positive measure* (i.e., sets which do not have measure zero).

We shall later construct forcing extensions $M[G]$ which are obtained by iterations of forcing notions similar to the above example. We shall require that in the iteration $M_{\alpha+1}$ is a generic extension of M_α by some forcing $Q_\alpha \in M_\alpha = M[G_\alpha]$; the forcing is in general only given by a *name* $\dot{Q}_\alpha \in M$ such that $Q_\alpha = \dot{Q}_\alpha^{G_\alpha}$. To ensure that this is always a partial order we also require that $1_{P_\alpha} \Vdash \dot{Q}_\alpha$ is a partial order. Technical details will be given later.

A principal idea is to let $\dot{Q}_\alpha$ be some canonical name for a partial order forcing a certain property to hold, like making the set of reals constructed so far a measure zero set. A central concern for such iterations, like for many forcings, is the preservation of cardinals.

3 Forcing axioms

The argument that the set $\mathbb{R}\cap M$ of ground model reals has measure zero in the standard Cohen extension $M[H]=M[c]$ by the Cohen partial order Q rests, like most forcing arguments, on density considerations. For a given $\varepsilon=2^{-i}$, a sequence $I_0, I_1, I_2, \ldots$ of real intervals such that $\sum_{n<\omega}\text{length}(I_n)\leqslant\varepsilon$ is extracted from the Cohen real c. It remains to show that $X\subseteq\bigcup_{n<\omega}I_n$. For $x\in\mathbb{R}\cap M$ a dense set D_x is defined so that $H\cap D_x\neq\emptyset$ implies that $x\in\bigcup_{n<\omega}I_n$. To cover the real x requires a "partially generic filter" which intersects D_x. This approach is captured by the following definition:

Definition 3. *Let $(Q,\leqslant,1_Q)$ be a forcing, $\mathcal{D}$ be any set, and κ a cardinal.*

a) *$H\subseteq Q$ is a* filter *on Q if $\forall p\in H\forall q\in Q\,(p\leqslant q\rightarrow q\in H)$ and $\forall p,q\in H\exists r\in H\,(r\leqslant p\wedge r\leqslant q)$.*

b) *A filter H on Q is* $\mathcal{D}$-generic *if $D\cap G\neq\emptyset$ for every $D\in\mathcal{D}$ which is dense in Q.*

c) *The* forcing axiom $\mathrm{FA}_\kappa(Q)$ *postulates that there exists a* $\mathcal{D}$-generic *filter on Q for any $\mathcal{D}$ of cardinality $\leqslant\kappa$.*

For any countable $\mathcal{D}$ we obtain the existence of generic filters just like in the case of ground models.

Theorem 4. (Rasiowa-Sikorski) $\mathrm{FA}_{\aleph_0}(Q)$ *holds for any partial order Q.*

Proof. Let $\mathcal{D}$ be countable. Take an enumeration $(D_n|n<\omega)$ of all $D\in\mathcal{D}$ which are dense in Q. Define an ω-sequence $q=q_0\geqslant q_1\geqslant q_2\geqslant\ldots$ recursively, using the axiom of choice:

$$\text{choose } q_{n+1} \text{ such that } q_{n+1}\leqslant q_n \text{ and } q_{n+1}\in D_n.$$

Then $H=\{q\in Q|\exists n<\omega\; q_n\leqslant q\}$ is as desired. □

Exercise 5. Show that $\mathrm{FA}_\kappa(Q)$ holds for any κ-closed partial order Q.

The results of the previous chapter now read as follows:

Theorem 5. *Let $Q=\mathrm{Fn}(\omega,2,\aleph_0)$ be the Cohen partial order and assume $\mathrm{FA}_{\aleph_1}(Q)$. Then every set of reals of cardinality $\leqslant\aleph_1$ has measure zero.*

Theorem 6. *Let $M[G]$ be a generic extension of the ground model M by λ-fold Cohen forcing $P=(P,\leqslant,1)=\mathrm{Fn}(\lambda\times\omega,2,\aleph_0)$ where $\lambda=\aleph_2^M$. Then in $M[G]$, $\mathrm{FA}_{\aleph_1}(Q)$ holds.*

Proof. We may assume that every $D\in\mathcal{D}$ is a dense subset of Q. Then $\mathcal{D}$ can be coded as a subset of $\aleph_1^M$. There is $\alpha<\lambda$ such that $\mathcal{D}\in M[G_\alpha]$. The filter H_α corresponding to the α-th Cohen real in the construction is $M[G_\alpha]$-generic on Q. Since $\mathcal{D}\subseteq M[G_\alpha]$, H_α is $\mathcal{D}$-generic on Q. □

So for the Cohen forcing Q we have a strengthening of the Rasiowa-Sikorski Lemma from countable to cardinality $\leqslant\aleph_1$. This is not possible for all forcings:

Lemma 7. *Let $P=\mathrm{Fn}(\aleph_0,\aleph_1,\aleph_0)$ be the canonical forcing for adding a surjection from $\aleph_0$ onto $\aleph_1$. Then $\mathrm{FA}_{\aleph_1}(P)$ is false.*

Proof. For $\alpha<\aleph_1$ define the set

$$D_\alpha=\{p\in P\mid \alpha\in\mathrm{ran}(p)\}$$

which is dense in P. Let $D=\{D_\alpha\mid\alpha<\aleph_1\}$. Assume for a contradiction that H is a $\mathcal{D}$-generic filter on P. Then $\bigcup H$ is a partial function from $\aleph_0$ to $\aleph_1$.

(1) $\bigcup H$ is onto $\aleph_1$.

Proof. Let $\alpha<\aleph_1$. Since H is a $\mathcal{D}$-generic, $H\cap D_\alpha\neq\emptyset$. Take $p\in H\cap D_\alpha$. Then

$$\alpha\in\mathrm{ran}(p)\subseteq\mathrm{ran}\Big(\bigcup H\Big)$$

qed.

But this is a contradiction since $\aleph_1$ is a cardinal. □

Exercise 6. Show that $\mathrm{FA}_{2^{\aleph_0}}(\mathrm{Fn}(\aleph_0,\aleph_0,\aleph_0))$ is false.

So we cannot have an uncountable generalization of the Rasiowa-Sikorski Lemma for forcings which collapse the cardinal $\aleph_1$. Since countable chain condition (ccc) forcing does not collapse cardinals, this suggests the following axiom:

Definition 8.

a) *Let κ be a cardinal. Then* Martin's axiom MA_κ *is the property: for every ccc partial order $(P,\leqslant,1_P)$, $\mathrm{FA}_\kappa(P)$ holds.*

b) Martin's axiom MA *postulates that MA_κ holds for every $\kappa<2^{\aleph_0}$.*

$\mathrm{MA}_{\aleph_0}$ holds by Theorem 4. Thus the continuum hypothesis $2^{\aleph_0}=\aleph_1$ trivially implies MA. We shall later see by an iterated forcing construction that $2^{\aleph_0}=\aleph_2$ and MA are relatively consistent with ZFC.

4 Consequences of MA+¬CH

4.1 Lebesgue measure

We shall not go into the details of LEBESGUE measure, since we shall only consider measure zero sets. We recall some notions and facts from before. For $s \in {}^{<\omega}2 = \{t | t\colon \mathrm{dom}(t) \to 2 \wedge \mathrm{dom}(t) \in \omega\}$ define the real *interval*

$$I_s = \{x \in \mathbb{R} | s \subseteq x\} \subseteq \mathbb{R}$$

with $\mathrm{length}(I_s) = 2^{-\mathrm{dom}(s)}$. Note that $I_s = I_{s \cup \{(\mathrm{dom}(s),0)\}} \cup I_{s \cup \{(\mathrm{dom}(s),1)\}}$, $\mathrm{length}(\mathbb{R}) = I_\emptyset = 2^{-0} = 1$, and $\mathrm{length}(I_{s \cup \{(\mathrm{dom}(s),0)\}}) = \mathrm{length}(I_{s \cup \{(\mathrm{dom}(s),1)\}}) = \frac{1}{2} \mathrm{length}(I_s)$.

Definition 9. *Let $\varepsilon > 0$. Then a set $X \subseteq \mathbb{R}$ has* measure $<\varepsilon$ *if there exists a sequence $(I_n | n < \omega)$ of intervals in $\mathbb{R}$ such that $X \subseteq \bigcup_{n<\omega} I_n$ and $\sum_{n<\omega} \mathrm{length}(I_n) \leqslant \varepsilon$. A set $X \subseteq \mathbb{R}$ has* measure zero *if it has measure $<\varepsilon$ for every $\varepsilon > 0$.*

The measure zero sets form a countably complete ideal on $\mathbb{R}$. It is easy to see that a countable union of measure zero sets is again measure zero. To strengthen this theorem in the context of MA we need some more topological and measure theoretic notions. The (standard) topology on $\mathbb{R}$ is generated by the basic open sets I_s for $s \in {}^{<\omega}2$. Hence every union $\bigcup_{n<\omega} I_n$ of basic open intervals is itself open. The basic open intervals I_s are also compact in the sense of the HEINE-BOREL theorem: every cover of I_s by open sets has a finite subcover.

Theorem 10. *Assume MA_κ and let $(X_i | i < \kappa)$ be a family of measure zero sets. Then $X = \bigcup_{i<\kappa} X_i$ has measure zero.*

Proof. Fix $\varepsilon > 0$. We show that $X = \bigcup_{i<\kappa} X_i$ has measure $<2\varepsilon$. Let

$$\mathcal{I} = \{(a,b) | a, b \in \mathbb{Q}, a < b\}$$

the countable set of rational intervals $(a,b) = \{c \in \mathbb{R} | a < c < b\}$ in $\mathbb{R}$. The *length* of (a,b) is simply $\mathrm{length}((a,b)) = b - a$. We shall apply MARTIN's axiom to the following forcing $P = (P, \supseteq, \emptyset)$ where

$$P = \{p \subseteq \mathcal{I} | \sum_{I \in p} \mathrm{length}(I) < \varepsilon\}.$$

(1) P is ccc.
Proof. Let $\{p_i | i < \omega_1\} \subseteq P$. For every $i < \omega_1$ there is $n_i < \omega$ such that p_i has measure $<\varepsilon - \frac{1}{n_i}$. By a pigeonhole principle we may assume that all n_i are equal to a common value $n < \omega$. For every p_i we have

$$\sum_{I \in p_i} \mathrm{length}(I) < \varepsilon - \frac{1}{n}.$$

For every $i<\omega_1$ take a finite set $\bar{p}_i \subseteq p_i$ such that

$$\sum_{I\in p_i\setminus\bar{p}_i} \operatorname{length}(I) < \frac{1}{n}.$$

There are only countably many such set $\bar{p}_i$, and again by a pigeonhole argument we may assume that for all $i<\omega_1$

$$\bar{p}_i = \bar{p}$$

takes a fixed value. Now consider $i<j<\omega_1$. Then

$$\begin{aligned}\sum_{I\in p_i\cup p_j} \operatorname{length}(I) &\leqslant \sum_{I\in p_i} \operatorname{length}(I) + \sum_{I\in p_j\setminus\bar{p}} \operatorname{length}(I)\\ &< \varepsilon - \frac{1}{n} + \frac{1}{n}\\ &= \varepsilon\end{aligned}$$

Hence $p_i \cup p_j \in P$ and $p_i \cup p_j \leqslant p_i, p_j$, and so $\{p_i | i<\omega_1\}$ is not an antichain in P. *qed*(1)

For $i<\kappa$ define

$$D_i = \{p\in P | X_i \subseteq \bigcup p\}.$$

(2) D_i is dense in P.
Proof. Let $q\in P$. Take $n<\omega$ such that

$$\sum_{I\in q} \operatorname{length}(I) < \varepsilon - \frac{1}{n}.$$

Since X_i has measure zero, take $r\subseteq\mathcal{I}$ such that $X_i\subseteq\bigcup p$ and $\sum_{I\in r}\operatorname{length}(I)\leqslant\frac{1}{n}$. Then

$$X_i \subseteq \bigcup (q\cup r) \text{ and } \sum_{I\in q\cup r}\operatorname{length}(I) \leqslant \sum_{I\in q}\operatorname{length}(I) + \sum_{I\in r}\operatorname{length}(I) < \varepsilon - \frac{1}{n} + \frac{1}{n} = \varepsilon.$$

Hence $p = q\cup r\in P$, $p\supseteq q$, and $p\in D_i$. *qed*(2)

By MA_κ take a filter G on P which is $\{D_i | i<\kappa\}$-generic. Let $U = \bigcup G \subseteq \mathcal{I}$.

(3) $X = \bigcup_{i<\kappa} X_i \subseteq \bigcup_{I\in U} I$.
Proof. Let $i<\kappa$. By the generity of G take $p\in G\cap D_i$. Then

$$X_i \subseteq \bigcup p \subseteq \bigcup U$$

qed(3)

(4) $\sum_{I \in U} \text{length}(I) \leqslant \varepsilon$.
Proof. Assume for a contradiction that $\sum_{I \in U} \text{length}(I) > \varepsilon$. Then take a finite set $\bar{U} \subseteq U$ such that $\sum_{I \in \bar{U}} \text{length}(I) > \varepsilon$. Let $\bar{B} = \{I_0, \ldots, I_{k-1}\}$. For every $I_j \in \bar{U}$ take $p_j \in G$ such that $I_j \in p_j$. Since all elements of G are compatible within G there is a condition $p \in G$ such that $p \supseteq p_0, \ldots, p_{k-1}$. Hence $\bar{U} \subseteq p$. But, since $p \in P$, we get a contradiction:

$$\varepsilon < \sum_{I \in \bar{U}} \text{length}(I) \leqslant \sum_{I \in p} \text{length}(I) < \varepsilon.$$

□

Two easy consequences are:

Corollary 11. *Assume* MA_κ *and let* $X \subseteq \mathbb{R}$ *with* $\text{card}(X) \leqslant \kappa$. *Then* X *has measure zero.*

Theorem 12. *Assume* MA. *Then* $2^{\aleph_0}$ *is regular.*

Proof. Assume instead that $\mathbb{R} = \bigcup_{i<\kappa} X_i$ for some $\kappa < 2^{\aleph_0}$, where $\text{card}(X_i) < 2^{\aleph_0}$ for every $i < \kappa$. Every singleton $\{r\}$ has measure zero. By Theorem 10, each X_i has measure zero. Again by Theorem, $\mathbb{R} = \bigcup_{i<\kappa} X_i$ has measure zero. But measure theory (and also intuition) shows that $\mathbb{R}$ does not have measure zero. □

4.2 Almost disjoint forcing

We intend to code subsets of κ by subsets of ω. If such a coding is possible then we shall have

$$2^{\aleph_0} \leqslant 2^\kappa \leqslant 2^{\aleph_0}, \text{ i.e. } 2^\kappa = 2^{\aleph_0}.$$

We shall employ almost disjoint coding.

Definition 13. *A sequence* $(x_i | i \in I)$ *is* almost disjoint *if*

a) x_i *is infinite*

b) $i \neq j < \kappa$ *implies that* $x_i \cap x_j$ *is finite*

Lemma 14. *There is an almost disjoint sequence* $(x_i | i < 2^{\aleph_0})$ *of subsets of* ω.

Proof. For $u \in {}^\omega 2$ let $x_u = \{u \upharpoonright m \,|\, m < \omega\}$. x_u is infinite. Consider $u \neq v$ from ${}^\omega 2$. Let $n < \omega$ be minimal such that $u \upharpoonright n \neq v \upharpoonright n$. Then

$$x_u \cap x_v = \{u \upharpoonright m \,|\, m < \omega\} \cap \{v \upharpoonright m \,|\, m < \omega\} = \{u \upharpoonright m \,|\, m < n\}$$

is finite. Thus $(x_u|u \in {}^{\omega}2)$ is almost disjoint. Using bijections $\omega \leftrightarrow {}^{<\omega}2$ and $2^{\aleph_0} \leftrightarrow {}^{\omega}2$ one can turn this into an almost disjoint sequence $(x_i|i < 2^{\aleph_0})$ of subsets of ω. $\square$

Theorem 15. *Assume* MA_κ. *Then* $2^\kappa = 2^{\aleph_0}$.

Proof. By a previous example, $\kappa < 2^{\aleph_0}$. By the lemma, fix an almost disjoint sequence $(x_i|i < \kappa)$ of subsets of ω. Define a map $c: \mathcal{P}(\omega) \to \mathcal{P}(\kappa)$ by

$$c(x) = \{i < \kappa | x \cap x_i \text{ is infinite}\}.$$

We say that x *codes* $c(x)$. We want to show that every subset of κ can be coded as some $c(x)$. We show this by proving that $c: \mathcal{P}(\omega) \to \mathcal{P}(\kappa)$ is surjective.

Let $A \subseteq \kappa$ be given. We use the following forcing $(P, \leqslant, 1)$ to code A:

$$P = \{(a, z) | a \subseteq \omega, z \subseteq \kappa, \operatorname{card}(a) < \aleph_0, \operatorname{card}(z) < \aleph_0\},$$

partially ordered by

$$(a', z') \leqslant (a, z) \text{ iff } a' \supseteq a, z' \supseteq z, i \in z \cap (\kappa \setminus A) \to a' \cap x_i = a \cap x_i.$$

The weakest element of P is $1 = (\emptyset, \emptyset)$.

The idea of the forcing is to keep the intersection of the first component with x_i fixed, provided $i \notin A$ has entered the second component. This will allow the almost disjoint coding of A by the finite/infinite method.
(1) $(P, \leqslant, 1)$ satisfies ccc.
Proof. Conditions (a, y) and (a, z) with equal first components are compatible, since $(a, y \cup z) \leqslant (a, y)$ and $(a, y \cup z) \leqslant (a, z)$. Incompatibel conditions have different first components. Since there are only countably many first components, an antichain in P can be at most countable. *qed*(1)

The outcome of a forcing construction results from an interplay between the partial order and some dense set arguments. We now define dense sets for our requirements.

For $i < \kappa$ let $D_i = \{(a, z) \in P | i \in z\}$. D_i is obviously dense in P. For $i \in A$ and $n \in \omega$ let $D_{i,n} = \{(a, z) \in P | \exists m > n: m \in a \cap x_i\}$.
(2) If $i \in A$ and $n \in \omega$ then $D_{i,n}$ is dense in P.
Proof. Consider $(a, z) \in P$. For $j \in z$, $j \neq i$ is the intersection $x_i \cap x_j$ finite. Take some $m \in x_i$, $m > n$ such that $m \notin x_i \cap x_j$ for $j \in z$, $j \neq i$. Then

$$(a \cup \{m\}, z) \leqslant (a, z) \text{ and } (a \cup \{m\}, z) \in D_{i,n}.$$

qed(2)

By MA_κ take a filter G on P which is generic for the dense sets in

$$\{D_i | i < \kappa\} \cup \{D_{i,n} | i \in A, n \in \omega\}.$$

Let

$$x = \bigcup \{a | (a, y) \in G\} \subseteq \omega.$$

(3) Let $i \in A$. Then $x \cap x_i$ is infinite.
Proof. Let $n < \omega$. By genericity take $(a, y) \in G \cap D_{i,n}$. By the definition of $D_{i,n}$ take $m > n$ such that $m \in a \cap x_i$. Then $m \in x \cap x_i$, and so $x \cap x_i$ is cofinal in ω. *qed*(3)
(4) Let $i \in \kappa \setminus A$. Then $x \cap x_i$ is finite.
Proof. By genericity take $(a, y) \in G \cap D_i$. Then $i \in y$. We show that $x \cap x_i \subseteq a \cap x_i$. Consider $n \in x \cap x_i$. Take $(b, z) \in G$ such that $n \in b$. By the filter properties of G take $(a', y') \in P$ such that $(a', y') \leqslant (a, y)$ and $(a', y') \leqslant (b, z)$. Then $n \in a'$, and by the definition of $\leqslant$, $a' \cap x_i = a \cap x_i$. Thus $n \in a \cap x_i$. *qed*(4)

So

$$c(x) = \{i < \kappa | x \cap x_i \text{ is infinite}\} = A \in \mathrm{range}(c).$$

□

5 Iterated forcing

Martin's axiom postulates that for every ccc partial order $(P, \leqslant, 1_P)$ and $\mathcal{D}$ with $\mathrm{card}(\mathcal{D}) < 2^{\aleph_0}$ there is a $\mathcal{D}$-generic filter G on P. Syntactically this axiom has a $\forall\exists$-form: $\forall P \forall \mathcal{D} \exists G \ldots$. $\forall\exists$-properties are often realised through chain constructions: build a chain

$$M = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_\alpha \subseteq \ldots \subseteq M_\beta \subseteq \ldots$$

of models such that for any $P, \mathcal{D} \in M_\alpha$ there is some $\beta \geqslant \alpha$ such that M_β contains a generic G as required. Then the "union" or limit of the chain should contain appropriate G's for all P's and $\mathcal{D}$'s.

Such chain constructions are wellknown from algebra. To satisfy closure under square roots ($\forall x \exists y$: $yy = x$) one can e.g. start with a countable field M_0 and along a chain $M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots$ adjoin square roots for all elements of M_n. Then $\bigcup_{n<\omega} M_n$ satisfies the closure property.

In set theory there is a difficulty that unions of models of set theory usually do not satisfy the theory ZF: assume that $M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots$ is an ascending chain of transitive models of ZF such that $(M_{n+1} \setminus M_n) \cap \mathcal{P}(\omega) \neq \emptyset$ for all $n < \omega$. Let $M_\omega = \bigcup_{n<\omega} M_n$. Then $\mathcal{P}(\omega) \cap M_\omega \notin M_\omega$. Indeed, if one had $\mathcal{P}(\omega) \cap M_\omega \in M_\omega$ then $\mathcal{P}(\omega) \cap M_\omega \in M_n$ for some $n < \omega$ and $\mathcal{P}(\omega) \cap M_{n+1} \in M_n$ contradicts the initial assumption. So a "limit" model of models of ZF has to be more complicated, and it will itself be constructed by some limit forcing which is called *iterated forcing*.

Exercise 7. Check which axioms of set theory hold in $M_\omega = \bigcup_{n<\omega} M_n$ where $(M_n)_{n<\omega}$ is an ascending sequence of transitive models of ZF(C).

Since we want to obtain the limit by forcing over a ground model M the construction must be visible in the ground model. This means that the sequence of forcings to be employed to pass from M_α to $M_{\alpha+1}$ has to exist as a sequence $(\dot{Q}_\beta|\beta<\kappa)$ of names in the ground model. The initial sequence $(\dot{Q}_\beta|\beta<\alpha)$ already determines a forcing P_α and $\dot{Q}_\alpha$ is intended to be a P_α-name. If G_α is M-generic over P_α then furthermore $Q_\alpha=(\dot{Q}_\alpha)^{G_\alpha}$ is intended to be a forcing in the model $M_\alpha = M[G_\alpha]$, and $M_{\alpha+1}$ is a generic extension of M_α by forcing with Q_α. The following iteration theorem says that any sequence $(\dot{Q}_\beta|\beta<\kappa)\in M$ gives rise to an iteration of forcing extensions. In applications the sequence has to be chosen carefully to ensure that some $\forall\exists$-property holds in the final model M_κ. Without loss of generality we only consider forcings Q_α whose maximal element is $\emptyset$.

Theorem 16. *Let M be a ground model, and let $((\dot{Q}_\beta,\dot{\leqslant}_\beta)|\beta<\kappa)\in M$ with the property that $\forall\beta<\kappa\colon\emptyset\in\operatorname{dom}(\dot{Q}_\beta)$. Then there is a uniquely determined sequence $((P_\alpha,\leqslant_\alpha,1_\alpha)|\alpha\leqslant\kappa)\in M$ such that*

a) $(P_\alpha,\leqslant_\alpha,1_\alpha)$ is a partial order which consists of α-sequences;

b) $P_0=\{\emptyset\}$, $\leqslant_0=\{(\emptyset,\emptyset)\}$, $1_0=\emptyset$;

c) If $\lambda\leqslant\kappa$ is a limit ordinal then the forcing P_λ is defined by:

$$\begin{aligned} P_\lambda &= \{p\colon\lambda\to V\,|\,(\forall\gamma<\lambda\colon p\restriction\gamma\in P_\gamma)\wedge\exists\gamma<\lambda\forall\beta\in[\gamma,\lambda)\ p(\beta)=\emptyset)\}\\ p\leqslant_\lambda q &\text{ iff } \forall\gamma<\lambda\colon p\restriction\gamma\leqslant_\gamma q\restriction\gamma\\ 1_\lambda &= (\emptyset|\gamma<\lambda)\end{aligned}$$

d) If $\alpha<\kappa$ and $1_\alpha\Vdash_{P_\alpha}$ "$(\dot{Q}_\alpha,\dot{\leqslant}_\alpha,\emptyset)$ is a forcing", then the forcing $P_{\alpha+1}$ is defined by:

$$\begin{aligned} P_{\alpha+1} &= \{p\colon\alpha+1\to V\,|\,p\restriction\alpha\in P_\alpha\wedge p(\alpha)\in\operatorname{dom}(\dot{Q}_\alpha)\wedge p\restriction\alpha\Vdash_{P_\alpha}p(\alpha)\in\dot{Q}_\alpha\}\\ p\leqslant_{\alpha+1}q &\text{ iff } p\restriction\alpha\leqslant_\alpha q\restriction\alpha\wedge p\restriction\alpha\Vdash_{P_\alpha}p(\alpha)\dot{\leqslant}_\alpha q(\alpha)\\ 1_{\alpha+1} &= (\emptyset|\gamma<\alpha+1)\end{aligned}$$

e) If $\alpha<\kappa$ and not *$1_\alpha\Vdash_{P_\alpha}$ "$(\dot{Q}_\alpha,\dot{\leqslant}_\alpha,\emptyset)$ is a forcing", then the forcing $P_{\alpha+1}$ is defined by:*

$$\begin{aligned} P_{\alpha+1} &= \{p\colon\alpha+1\to V\,|\,p\restriction\alpha\in P_\alpha\wedge p(\alpha)=\emptyset\}\\ p\leqslant_{\alpha+1}q &\text{ iff } p\restriction\alpha\leqslant_\alpha q\restriction\alpha\\ 1_{\alpha+1} &= (\emptyset|\gamma<\alpha+1)\end{aligned}$$

$((P_\alpha,\leqslant_\alpha,1_\alpha)|\alpha\leqslant\kappa)$, and in particular P_κ are called the (*finite support*) *iteration* of the sequence $((\dot{Q}_\beta,\dot{\leqslant}_\beta)|\beta<\kappa)$.

Proof. To justify the above recursive definition of the sequence $((P_\alpha,\leqslant_\alpha,1_\alpha)|\alpha\leqslant\kappa)$ it suffices to show recursively that every P_α is a forcing.

Obviously, P_0 is a trivial one-element forcing.

Consider a limit $\lambda \leqslant \kappa$ and assume that P_γ is a forcing for $\gamma < \alpha$. We have to show that the relation $\leqslant_\lambda$ is transitive with maximal element 1_λ. Consider $p \leqslant_\lambda q \leqslant_\lambda r$. Then $\forall \gamma < \lambda$: $p \upharpoonright \gamma \leqslant_\gamma q \upharpoonright \gamma$ and $\forall \gamma < \lambda$: $q \upharpoonright \gamma \leqslant_\gamma r \upharpoonright \gamma$. Since all $\leqslant_\gamma$ with $\gamma < \lambda$ are transitive relations, $\forall \gamma < \lambda$: $p \upharpoonright \gamma \leqslant_\gamma r \upharpoonright \gamma$ and so $p \leqslant_\lambda r$. Now consider $p \in P_\lambda$. Then $\forall \gamma < \lambda$: $p \upharpoonright \gamma \in P_\gamma$. By the inductive assumption, $\forall \gamma < \lambda$: $p \upharpoonright \gamma \leqslant_\gamma 1_\gamma = 1_\lambda \upharpoonright \gamma$ and so $p \leqslant_\lambda 1_\lambda$.

For the successor step assume that $\alpha < \kappa$ and that P_α is a forcing.
Case 1. $1_\alpha \Vdash_{P_\alpha} (\dot{Q}_\alpha, \dot{\leqslant}_\alpha, \emptyset)$ is a forcing.
For the transitivity of $\leqslant_{\alpha+1}$ consider $p \leqslant_{\alpha+1} q \leqslant_{\alpha+1} r$. Then $p \upharpoonright \alpha \leqslant_\alpha q \upharpoonright \alpha \wedge p \upharpoonright \alpha \Vdash_{P_\alpha} p(\alpha) \dot{\leqslant}_\alpha q(\alpha)$ and $q \upharpoonright \alpha \leqslant_\alpha r \upharpoonright \alpha \wedge q \upharpoonright \alpha \Vdash_{P_\alpha} q(\alpha) \dot{\leqslant}_\alpha r(\alpha)$. By the transitivity of $\leqslant_\alpha$: $p \upharpoonright \alpha \leqslant_\alpha r \upharpoonright \alpha$. Moreover $p \upharpoonright \alpha \Vdash_{P_\alpha} p(\alpha) \dot{\leqslant}_\alpha q(\alpha)$, $p \upharpoonright \alpha \Vdash_{P_\alpha} q(\alpha) \dot{\leqslant}_\alpha r(\alpha)$ and $p \upharpoonright \alpha \Vdash_{P_\alpha}$ "$\dot{\leqslant}_\alpha$ is transitive". This implies $p \upharpoonright \alpha \Vdash_{P_\alpha} p(\alpha) \dot{\leqslant}_\alpha r(\alpha)$ and together that $p \leqslant_{\alpha+1} r$.

For the maximality of $1_{\alpha+1}$ consider $p \in P_{\alpha+1}$. Then $p \upharpoonright \alpha \in P_\alpha \wedge p \upharpoonright \alpha \Vdash_{P_\alpha} p(\alpha) \in \dot{Q}_\alpha$. Then $p \upharpoonright \alpha \leqslant_\alpha 1_\alpha = 1_{\alpha+1} \upharpoonright \alpha$. Moreover $p \upharpoonright \alpha \Vdash_{P_\alpha}$ "$\emptyset$ is maximal in $\dot{\leqslant}_\alpha$" implies that $p \upharpoonright \alpha \Vdash_{P_\alpha} p(\alpha) \dot{\leqslant}_\alpha \emptyset = 1_{\alpha+1}(\alpha)$. Hence $p \leqslant_{\alpha+1} 1_{\alpha+1}$.
Case 2. It is not the case that $1_\alpha \Vdash_{P_\alpha} (\dot{Q}_\alpha, \dot{\leqslant}_\alpha, \emptyset)$ is a forcing.
For the transitivity of $\leqslant_{\alpha+1}$ consider $p \leqslant_{\alpha+1} q \leqslant_{\alpha+1} r$. Then $p \upharpoonright \alpha \leqslant_\alpha q \upharpoonright \alpha$ and $q \upharpoonright \alpha \leqslant_\alpha r \upharpoonright \alpha$. By the transitivity of $\leqslant_\alpha$: $p \upharpoonright \alpha \leqslant_\alpha r \upharpoonright \alpha$ and so $p \leqslant_{\alpha+1} r$.

For the maximality of $1_{\alpha+1}$ consider $p \in P_{\alpha+1}$. Then $p \upharpoonright \alpha \in P_\alpha$. By induction, $p \upharpoonright \alpha \leqslant_\alpha 1_\alpha$ and so $p \leqslant_{\alpha+1} 1_{\alpha+1}$. □

The term "finite support iteration" is justified by the following

Lemma 17. *In the above situation let $p \in P_\kappa$. Then*

$$\mathrm{supp}(p) = \{\alpha < \kappa \mid p(\alpha) \neq \emptyset\}$$

is finite.

Proof. Prove by induction on $\alpha \leqslant \kappa$ that $\mathrm{supp}(p)$ is finite for every $q \in P_\alpha$. The crucial property is the definition of P_λ at limit λ in the above iteration theorem. □

Let us fix a ground model M and the iteration $((\dot{Q}_\beta, \dot{\leqslant}_\beta) \mid \beta < \kappa) \in M$ and $((P_\alpha, \leqslant_\alpha, 1_\alpha) \mid \alpha \leqslant \kappa) \in M$ as above. Let G_κ be M-generic for P_κ. We analyse the generic extension $M_\kappa = M[G_\kappa]$ by an ascending chain

$$M = M_0 \subseteq M_1 = M[G_1] = M_0[H_0] \subseteq M_2 = M[G_2] = M_1[H_1] \subseteq \ldots \subseteq M_\alpha = M[G_\alpha] \subseteq \ldots \subseteq M_\kappa$$

of generic extensions.

Let us first note some relations within the tower $(P_\alpha)_{\alpha \leqslant \kappa}$ of forcings.

Lemma 18.

a) *Let* $\alpha \leqslant \kappa$ *and* $p, q \in P_\alpha$.
Then $p \leqslant_\alpha q$ *iff* $\forall \gamma \in \operatorname{supp}(p) \cup \operatorname{supp}(q)\colon p \upharpoonright \gamma \Vdash_{P_\gamma} p(\gamma) \dot{\leqslant}_\gamma q(\gamma)$.

b) *Let* $\alpha \leqslant \beta \leqslant \kappa$ *and* $p \in P_\beta$. *Then* $p \upharpoonright \alpha \in P_\alpha$.

c) *Let* $\alpha \leqslant \beta \leqslant \kappa$ *and* $p \leqslant_\beta q$. *Then* $p \upharpoonright \alpha \leqslant_\alpha q \upharpoonright \alpha$.

d) *Let* $\alpha \leqslant \beta \leqslant \kappa$, $q \in P_\beta$, $\bar{p} \leqslant_\alpha q \upharpoonright \alpha$. *Then* $\bar{p} \cup (q(\gamma) | \alpha \leqslant \gamma < \beta) \in P_\beta$ *and* $\bar{p} \cup (q(\gamma) | \alpha \leqslant \gamma < \beta) \leqslant_\beta q$.

Proof. *a*) By a straightforward induction on $\alpha \leqslant \kappa$. Now *b*) $-$ *d*) follow immediately. $\square$

For $\alpha \leqslant \kappa$ define $G_\alpha = \{p \upharpoonright \alpha \,|\, p \in G_\kappa\}$.
(1) G_α is M-generic for P_α.
Proof. By (a), $G_\alpha \subseteq P_\alpha$. Consider $p \upharpoonright \alpha, q \upharpoonright \alpha \in G_\alpha$ with $p, q \in G_\kappa$. Take $r \in G_\kappa$ such that $r \leqslant_\kappa p, q$. By (a), $r \upharpoonright \alpha \leqslant_\alpha p \upharpoonright \alpha, q \upharpoonright \alpha$. Thus all elements of G_α are compatible in P_α.

Consider $p \upharpoonright \alpha \in G_\alpha$ with $p \in G_\kappa$ and $\bar{q} \in P_\alpha$ with $p \upharpoonright \alpha \leqslant_\alpha \bar{q}$. By (a),

$$q = \bar{q} \cup (\emptyset | \alpha \leqslant \gamma < \kappa)$$

is an element of P_κ and $p \leqslant_\kappa q$. Since G_κ is a filter, $q \in G_\kappa$, and so $\bar{q} = q \upharpoonright \alpha \in G_\alpha$. Thus G_α is upwards closed.

For the genericity consider a set $\bar{D} \in M$ which is dense in P_α. We claim that the set

$$D = \{d \in P_\kappa \,|\, d \upharpoonright \alpha \in \bar{D}\} \in M$$

is dense in P_κ: let $p \in P_\kappa$. Then $p \upharpoonright \alpha \in P_\alpha$. Take $\bar{d} \in \bar{D}$ such that $\bar{d} \leqslant_\alpha p \upharpoonright \alpha$. By (c,d),

$$d = \bar{d} \cup (p(\gamma) | \alpha \leqslant \gamma < \kappa) \in P_\kappa$$

and $d \leqslant_\kappa p$.

By the genericity of G_κ take $p \in D \cap G_\kappa$. Then $p \upharpoonright \alpha \in \bar{D} \cap G_\alpha \neq \emptyset$. *qed*(1)

So $M_\alpha = M[G_\alpha]$ is a welldefined generic extension of M by G_α.
(2) Let $\alpha < \beta \leqslant \kappa$. Then $G_\alpha \in M[G_\beta]$ and $M[G_\alpha] \subseteq M[G_\beta]$.
Proof. $G_\alpha = \{p \upharpoonright \alpha \,|\, p \in G_\kappa\} = \{(p \upharpoonright \beta) \upharpoonright \alpha \,|\, p \in G_\kappa\} = \{q \upharpoonright \alpha \,|\, q \in G_\beta\} \in M[G_\beta]$. *qed*(2)

For $\alpha < \kappa$ define

$$Q_\alpha = (Q_\alpha, \leqslant^{Q_\alpha}, \emptyset) = \begin{cases} (\dot{Q}_\alpha^{G_\alpha}, \dot{\leqslant}_\alpha^{G_\alpha}, \emptyset), \text{ if } 1_\alpha \Vdash_{P_\alpha} \text{“}(\dot{Q}_\alpha, \dot{\leqslant}_\alpha, \emptyset) \text{ is a forcing”} \\ (\{\emptyset\}, \{(\emptyset, \emptyset)\}, \emptyset), \text{ else} \end{cases}$$

Then $Q_\alpha \in M_\alpha = M[G_\alpha]$ is a forcing. For $\alpha < \kappa$ define

$$H_\alpha = \{p(\alpha)^{G_\alpha} | p \in G_\kappa\}.$$

(3) H_α is M_α-generic for Q_α.
Proof. If it is not the case that $1_\alpha \Vdash_{P_\alpha}$ “$(\dot{Q}_\alpha, \dot{\leqslant}_\alpha, \emptyset)$ is a forcing”, then $(Q_\alpha, \leqslant^{Q_\alpha}, \emptyset) = (\{\emptyset\}, \{(\emptyset, \emptyset)\}, \emptyset)$ and $H_\alpha = \{\emptyset\}$ is trivially M_α-generic. So assume that $1_\alpha \Vdash_{P_\alpha}$ “$(\dot{Q}_\alpha, \dot{\leqslant}_\alpha, \emptyset)$ is a forcing”.
(a) $H_\alpha \subseteq Q_\alpha$.
Proof. Let $p \in G_\kappa$. Then $p \upharpoonright \alpha + 1 \in P_{\alpha+1}$ and so $p \upharpoonright \alpha \Vdash_{P_\alpha} p(\alpha) \in \dot{Q}_\alpha$. Since $p \upharpoonright \alpha \in G_\alpha$ we have that $p(\alpha)^{G_\alpha} \in \dot{Q}_\alpha^{G_\alpha} = Q_\alpha$. *qed*(a)
(b) H_α is a filter.
Proof. Straightforward.
(e) Let $\bar{D} \in M_\alpha$ be dense in Q_α. Then $\bar{D} \cap H_\alpha \neq \emptyset$.
Proof. Take $\dot{D} \in M$ such that $\bar{D} = \dot{D}^{G_\alpha}$. Take $p \in G_\kappa$ such that

$$p \upharpoonright \alpha \Vdash_{P_\alpha} \dot{D} \text{ is dense in } \dot{Q}_\alpha.$$

Define

$$D = \{d \in P_\kappa \,|\, d \upharpoonright \alpha \Vdash d(\alpha) \in \dot{D}\} \in M.$$

We show that D is dense in P_κ below p. Let $q \leqslant_\kappa p$. Then $q \upharpoonright \alpha \leqslant_\alpha p \upharpoonright \alpha$ and $q \upharpoonright \alpha \Vdash q(\alpha) \dot{\leqslant}_\alpha p(\alpha)$. Hence $q \upharpoonright \alpha \Vdash_{P_\alpha} \dot{D}$ is dense in $\dot{Q}_\alpha$ and there is $\bar{d} \leqslant_\alpha q \upharpoonright \alpha$ and some $d(\alpha) \in \mathrm{dom}(\dot{Q}_\alpha)$ such that

$$\bar{d} \Vdash_{P_\alpha} (d(\alpha) \dot{\leqslant}_\alpha q(\alpha) \wedge d(\alpha) \in \dot{D}).$$

Define

$$d = \bar{d} \cup \{(\alpha, d(\alpha))\} \cup (q(\gamma) | \alpha < \gamma < \kappa).$$

Then $d \in P_\kappa$, $d \leqslant_\kappa q$, and $d \in D$.

By the genericity of G_κ take $d \in D \cap G_\kappa$. Then $d(\alpha)^{G_\alpha} \in H_\alpha$, $d \upharpoonright \alpha \in G_\alpha$, and $d(\alpha)^{G_\alpha} \in (\dot{D})^{G_\alpha} = \bar{D}$. Thus $H_\alpha \cap \bar{D} \neq \emptyset$.
(4) $M_{\alpha+1} = M_\alpha[H_\alpha]$.
Proof. $\supseteq$ is straightforward. For the other direction, if suffices to show that $G_{\alpha+1} \in M_\alpha[H_\alpha]$, and indeed we show that

$$G_{\alpha+1} = \{q \in P_{\alpha+1} \,|\, q \upharpoonright \alpha \in G_\alpha \wedge q(\alpha)^{G_\alpha} \in H_\alpha\}.$$

Let $q \in G_{\alpha+1}$. Take $p \in G_\kappa$ such that $p \upharpoonright \alpha + 1 = q$. Then $q \upharpoonright \alpha = p \upharpoonright \alpha \in G_\alpha$ and $q(\alpha)^{G_\alpha} = p(\alpha)^{G_\alpha} \in H_\alpha$. For the converse consider $q \in P_{\alpha+1}$ such that $q \upharpoonright \alpha \in G_\alpha$ and $q(\alpha)^{G_\alpha} \in H_\alpha$. Take $p_1, p_2 \in G_\kappa$ such that $q \upharpoonright \alpha = p_1 \upharpoonright \alpha$ and $q(\alpha)^{G_\alpha} = p_2(\alpha)^{G_\alpha}$. Take $p \in G_\kappa$ such that $p \leqslant_\kappa p_1, p_2$. We also may assume that $p \upharpoonright \alpha \Vdash q(\alpha) = p_2(\alpha)$. $p \upharpoonright \alpha \leqslant_\alpha p_1 \upharpoonright \alpha = q \upharpoonright \alpha$ and $p \upharpoonright \alpha \Vdash_{P_\alpha} p(\alpha) \dot{\leqslant}_\alpha p_2(\alpha) = q(\alpha)$. Hence $p \upharpoonright \alpha + 1 \leqslant_{\alpha+1} q$. Since $p \upharpoonright \alpha + 1 \in G_{\alpha+1}$ and since $G_{\alpha+1}$ is upward closed, we get $q \in G_{\alpha+1}$.

5.1 Embeddings

In the above construction, $M[G_\alpha] \subseteq M[G_\beta]$ canonically. This corresponds to canonical transformations of names used in the construction of $M[G_\alpha]$ into names used to construct $M[G_\beta]$. Such transformation of names is important for the construction and analysis of interations. We first reduce our "name spaces" from all of M to more specific P-names.

Definition 19. *Let P be a forcing. Define recursively: $\dot{x}$ is a P-name if every element of $\dot{x}$ is an ordered pair $(\dot{y}, p)$ where $\dot{y}$ is a P-name and $p \in P$. Let V^P be the class or name space of all P-names.*

The generic interpretation of an arbitrary name only depends on ordered pairs whose second component is in P. This observation leads to

Lemma 20. *Let P be a forcing. Define $\tau: V \to V^P$ recursively by*

$$\tau(\dot{x}) = \{(\tau(\dot{y}), p) | (\dot{y}, p) \in \dot{x}\}.$$

Then $\tau(\dot{x})$ is a P-name and

$$1_P \Vdash \dot{x} = \tau(\dot{x}).$$

I.e., $\dot{x}^G = (\tau(\dot{x}))^G$ for every generic filter on P.

Let $\pi: P \to Q$ be an orderpreserving embedding of forcings. This induces an embedding of name spaces $\pi^*: V^P \to V^Q$ which is defined recursively:

$$\pi^*(\dot{x}) = \{(\pi^*(\dot{y}), \pi(p)) | (\dot{y}, p) \in \dot{x}\}.$$

One can study such embeddings in general. They satisfy "universal properties", sometimes relying on structural properties of the embedding π.

Exercise 8. Examine, how generic filters are mapped by π and its inverse and how this induces embeddings of generic extensions. Formulate sufficient properties for the original map π.

We restrict our considerations to embeddings connected to iterated forcing. So let $((P_\alpha, \leqslant_\alpha, 1_\alpha) | \alpha \leqslant \kappa)$ be a finite support iteration of the sequence $((\dot{Q}_\alpha, \dot{\leqslant}_\alpha) | \alpha < \kappa)$. In view of the previous lemma we also require in the iteration that every $\dot{Q}_\alpha$ is a P_α-name.

There are canonical maps between the P_α's. For $\alpha \leqslant \beta \leqslant \kappa$ define $\pi_{\alpha\beta}: P_\alpha \to P_\beta$ by

$$\pi_{\alpha\beta}(p) = p \cup (\emptyset \mid \alpha \leqslant \gamma < \beta).$$

Also define $\pi_{\beta\alpha}: P_\beta \to P_\alpha$ by $\pi_{\beta\alpha}(q) = q \upharpoonright \alpha$. $\pi_{\beta\alpha}$ is a left inverse of $\pi_{\alpha\beta}$:

$$\pi_{\beta\alpha} \circ \pi_{\alpha\beta} = \mathrm{id}_{P_\alpha}.$$

Let the previous constructions take place within a ground model M. Let G_κ be M-generic for P_κ and let $M_\alpha = M[G_\alpha]$ for $\alpha \leqslant \kappa$ be the associated tower of extensions. Let $\alpha \leqslant \beta \leqslant \kappa$. The inclusion $M[G_\alpha] \subseteq M[G_\beta]$ corresponds to the following

Lemma 21. *Let $\dot{x} \in M^{P_\alpha}$ be a P_α-name and $\ddot{x} = \pi^*_{\alpha\beta}(\dot{x}) \in M^{P_\beta}$ its "lift" to P_β. Then*

$$\dot{x}^{G_\alpha} = \ddot{x}^{G_\beta}.$$

Proof. By induction on $\dot{x}$:

$$\begin{aligned}
\ddot{x}^{G_\beta} &= \{\ddot{y}^{G_\beta} \mid \exists q \in G_\beta\, (\ddot{y}, q) \in \ddot{x}\} \\
&= \{\ddot{y}^{G_\beta} \mid \exists q\, (q \in G_\beta \wedge (\ddot{y}, q) \in \ddot{x})\} \\
&= \{\ddot{y}^{G_\beta} \mid \exists q\, (q \in G_\beta \wedge \exists (\dot{y}, p) \in \dot{x}\, ((\pi^*_{\alpha\beta}(\dot{y}), \pi_{\alpha\beta}(p)) \in \ddot{x} \wedge \ddot{y} = \pi^*_{\alpha\beta}(\dot{y}) \wedge q = \pi_{\alpha\beta}(p)))\} \\
&= \{\ddot{y}^{G_\beta} \mid \exists p \in G_\alpha\, \exists (\dot{y}, p) \in \dot{x}\ \ddot{y} = \pi^*_{\alpha\beta}(\dot{y})\} \\
&= \{\pi^*_{\alpha\beta}(\dot{y})^{G_\beta} \mid \exists p \in G_\alpha\, (\dot{y}, p) \in \dot{x}\} \\
&= \{\dot{y}^{G_\alpha} \mid \exists p \in G_\alpha\, (\dot{y}, p) \in \dot{x}\} \\
&= \dot{x}^{G_\alpha}
\end{aligned}$$

□

In the intended applications of iterated forcing we shall usually be confronted at "time" α with several tasks which have to be dealt with "one by one" along the ordinal axis κ: there will be, e.g., two distinct partial orders $R, S \in M[G_\alpha]$ for which we want to adjoin generic filters. These have P_α-names $\dot{R}, \dot{S} \in M^{P_\alpha}$. In the iteration we may set $\dot{Q}_\alpha = \dot{R}$, but then we have to deal with $\dot{S}$ at some later "time" β. This will be possible by lifting $\dot{S}$ to a P_β-name: set $\dot{Q}_\beta = \pi^*_{\alpha\beta}(\dot{S})$. In the construction some "bookkeeping mechanism" will ensure that eventually all tasks will be looked after.

5.2 Two-step iterations

Definition 22. *Consider a forcing $(P, \leqslant_P, 0)$ and names $\dot{Q}$, $\dot{\leqslant}$ such that*

$$1_P \Vdash (\dot{Q}, \dot{\leqslant}, 0) \text{ is a forcing.}$$

and $0\in\mathrm{dom}(\dot{Q})$. *Then the* two-step iteration $(P*\dot{Q},\preccurlyeq,1)$ *is defined by:*

$$\begin{array}{rcl} P*\dot{Q} & = & \{(p,\dot{q})|p\in P\wedge\dot{q}\in\mathrm{dom}(\dot{Q})\wedge p\Vdash_P\dot{q}\in\dot{Q}\} \\ (p',\dot{q}')\preccurlyeq(p,\dot{q}) & \textit{iff} & p'\leqslant_P p\wedge p'\Vdash_P\dot{q}'\dot{\leqslant}\dot{q}' \\ 1 & = & (0,0) \end{array}$$

The two-step iteration can be construed as an iteration of a sequence $((\dot{Q}_\beta,\dot{\leqslant}_\beta)|\beta<2)$ of length 2: Let $\dot{Q}_0=\check{P}$, $\dot{\leqslant}_0=\check{\leqslant}_P$ where the canonical names $\check{P}$ and $\check{\leqslant}_P$ are formed with respect to the trivial forcing $P_0=\{\emptyset\}$, $\leqslant_0=\{(\emptyset,\emptyset)\}$, $1_0=\emptyset$. Then $(P,\leqslant_P,0)$ is canonically isomorphic to the induced forcing $(P_1,\leqslant_1,1_1)$ by the map $h\colon p\longmapsto\check{p}$. We may assume that $\dot{Q}$ is a P-names in the restricted sense that for every ordered pair $(a,p)\in\mathrm{TC}(\dot{Q})$ $p\in P$. Then define a corresponding P_1-name $\dot{Q}_1$ by replacing recursively each $(a,p)\in\mathrm{TC}(\dot{Q})$ by $(...,h(p))$. Similarly for $\dot{\leqslant}_1$.

One can check that the iterated forcing $(P_2,\leqslant_2,1_2)$ defined from $((\dot{Q}_\beta,\dot{\leqslant}_\beta)|\beta<2)$ is canonically isomorphic to $(P*\dot{Q},\preccurlyeq,1)$.

Such identifications using subtle but canonical isomorphisms occur often in the theory of iterated forcing.

Exercise 9. If G is M-generic for $P*\dot{Q}\in M$ where M is a ground model define

$$\begin{array}{rcl} G_0 & = & \{p\in P\,|\,\exists\dot{q}\in\mathrm{dom}(\dot{Q})\colon(p,\dot{q})\in G\} \\ G_1 & = & \{\dot{q}^{G_0}\,|\,\exists p\in P\colon(p,\dot{q})\in G\} \end{array}$$

Show that G_0 is M-generic for P and that G_1 is M-generic for $\dot{Q}^{G_0}$.

Conversely let G_0 be M-generic for P and G_1 M-generic for $\dot{Q}^{G_0}$. Show that

$$G=\{(p,\dot{q})\,|\,p\in G_0,\dot{q}^{G_0}\in G_1\}$$

is M-generic for $P*\dot{Q}$.

5.3 Products of partial orders

A special case of a finite support iteration is a finite support *product*. So let M be a ground model, and let $((Q_\beta,\leqslant_\beta)|\beta<\kappa)\in M$ be a sequence of forcings such that $\emptyset$ is a maximal element of every Q_β. Define the *finite support product* $\prod_{\beta<\kappa}Q_\beta$ as the following forcing:

$$\begin{array}{rcl} \prod_{\beta<\kappa}Q_\beta & = & \{p\colon\kappa\to V\,|\forall\beta<\kappa\colon p(\beta)\in Q_\beta,\mathrm{supp}(p)\text{ is finite}\} \\ p\preccurlyeq q & \text{iff} & \forall\beta<\kappa\colon p(\beta)\leqslant_\beta q(\beta) \\ 1_\kappa & = & (0|\beta<\kappa) \end{array}$$

We want to show that the product corresponds to a simple iteration. Define a sequence

$$((\check{Q}_\beta, \check{\leqslant}_\beta) | \beta < \kappa) \in M$$

where $\check{Q}_\beta$ is the canonical name for Q_β with respect to a forcing which has the β-sequence $1_\beta = (0|\gamma < \beta)$ as its maximal element. (Note that the definition of $\check{x} = \{(\check{y}, 1_\beta) | y \in x\}$ only depends on 1_β .) Let the sequence $((P_\alpha, \leqslant_\alpha, 1_\alpha) | \alpha \leqslant \kappa) \in M$ be defined from the sequence $((\check{Q}_\beta, \check{\leqslant}_\beta) | \beta < \kappa)$ of names as in the iteration theorem.

Then there is a canonical isomorphism

$$\pi: \prod_{\beta < \kappa} Q_\beta \leftrightarrow P_\kappa$$

defined by: $p \mapsto p'$ where

$$p'(\beta) = \overbrace{p(\beta)}$$

with respect to a partial order with maximal element 1_β . It is tedious but straightforward to check that this defines an isomorphism.

6 Iteration theorems

A main concern of forcing is the preservation of cardinals. There are several criteria for ensuring cardinal preservation or at least the preservation of $\aleph_1$. Iteration theorems take the form: if every $\dot{Q}_\beta$ in $((\dot{Q}_\beta, \dot{\leqslant}_\beta) | \beta < \kappa)$ is forced to satisfy the preservation criterion then also P_κ satisfies the criterion.

Theorem 23. *Let λ be a regular cardinal. Consider the two-step iteration $(P * \dot{Q}, \preccurlyeq, 1)$ of $(P, \leqslant_P, 0)$ and $(\dot{Q}, \dot{\leqslant}, 0)$. Assume that $(P, \leqslant_P, 0)$ satisfies the λ-c.c. and $0 \Vdash_P$ "$(\dot{Q}, \dot{\leqslant}, 0)$ satisfies $\check{\lambda}$-c.c". Then $(P * \dot{Q}, \preccurlyeq, 1)$ satisfies the λ-c.c.*

Proof. We may assume that the assumptions of the theorem are satisfied in some ground model M. It suffices to prove the theorem in M. Work inside M. Let $((p_\alpha, q_\alpha) | \alpha < \lambda)$ be a sequence in $(P * \dot{Q}, \preccurlyeq, 1)$. It suffices to find two compatible conditions in this sequence.
(1) There is a condition $p \in P$ such that $p \Vdash \sup \{\alpha \mid \check{p}_\alpha \in \dot{G}\} = \lambda$ where $\dot{G}$ is the canonical name for a generic filter on P.
Proof. If not, then there is a maximal antichain A in P of conditions q for which there is an ordinal $\gamma_q < \kappa$ such that $q \Vdash \sup \{\alpha \mid \check{p}_\alpha \in \dot{G}\} = \check{\gamma}_q$. By the κ-c.c., card$(A) < \lambda$. By the regularity of κ there is $\gamma < \kappa$ such that $\forall q \in A \gamma_q < \gamma$. Since A is a maximal antichain,

$$0 = 1_P \Vdash \sup \{\alpha \mid \check{p}_\alpha \in \dot{G}\} \leqslant \check{\gamma}.$$

But $p_{\gamma+1} \Vdash \check{p}_{\gamma+1} \in \dot{G}$ and $p_{\gamma+1} \Vdash \sup\{\alpha \mid \check{p}_\alpha \in \dot{G}\} \geqslant \check{\gamma}+1$. Contradiction. *qed*(1)

Take an M-generic filter G on P such that $p \in G$ and $p \Vdash \sup\{\alpha \mid \check{p}_\alpha \in \dot{G}\} = \lambda$. In $M[G]$ form the sequence $(q_\alpha^G \mid p_\alpha \in G)$; by (1) this sequence has ordertype λ. $\dot{Q}^G$ satisfies the λ-c.c. in $M[G]$ and λ is still a regular cardinal in $M[G]$. So there are $\alpha < \beta < \lambda$ such that q_α^G and q_β^G are compatible in $\dot{Q}^G$. Take $r \in G$ and $q \in \mathrm{dom}(\dot{Q})$ such that $r \leqslant p_\alpha, p_\beta$ and $r \Vdash q \leqslant q_\alpha, q_\beta$. Then $(r, q) \in P * \dot{Q}$ and $(r, q) \preccurlyeq (p_\alpha, q_\alpha), (p_\beta, q_\beta)$. □

Theorem 24. *Let $((P_\alpha, \leqslant_\alpha, 1_\alpha) | \alpha \leqslant \kappa)$ be the finite support iteration of the sequence $((\dot{Q}_\beta, \dot{\leqslant}_\beta) | \beta < \kappa)$. Let λ be a regular cardinal and suppose that*

$$P_\beta \Vdash \text{“}\dot{Q}_\beta \text{ is } \check{\lambda}\text{-cc”}$$

for all $\beta < \kappa$. Then every P_α, $\alpha \leqslant \kappa$ is λ-cc.

Proof. Again it suffices to prove the theorem in some ground model M. Work inside M. We prove the theorem by induction on $\alpha \leqslant \kappa$. The theorem is trivial for $P_0 = \{\emptyset\}$.

Let $\alpha = \beta + 1$. One can canonically prove that $P_{\beta+1} \cong P_\beta * \dot{Q}_\beta$. Then P_α is λ-cc by the inductive assumption and the previous theorem.

Finally consider a limit ordinal $\alpha \leqslant \kappa$. Let $A \subseteq P_\alpha$ have cardinality λ. Every condition $p \in A$ has a finite support $\mathrm{supp}(p)$. By the Δ-system lemma, we may suppose that $(\mathrm{supp}(p) \mid p \in A)$ is a Δ-system with some finite kernel d. Take $\beta < \alpha$ such that $d \subseteq \beta$. By the inductive assumption P_β is λ-cc. Take distinct $p, q \in A$ such that $p \upharpoonright \beta, q \upharpoonright \beta$ are compatible in P_β. Take $r \in P_\beta$ such that $r \leqslant_\beta p \upharpoonright \beta, q \upharpoonright \beta$. We then define a compatibility element $s \leqslant_\alpha p, q$ by

$$s(i) = \begin{cases} r(i), \text{ for } i < \beta \\ p(i), \text{ for } \beta \leqslant i < \alpha, i \in \mathrm{supp}(p) \\ q(i), \text{ for } \beta \leqslant i < \alpha, i \notin \mathrm{supp}(p) \end{cases}$$

□

Although the final model $M[G_\kappa]$ is not the union of the models $M[G_\beta]$ it may behave like a union with respect to “small” sets.

Lemma 25. *In a ground model M let $((P_\alpha, \leqslant_\alpha, 1_\alpha) | \alpha \leqslant \kappa)$ be the finite support iteration of the sequence $((\dot{Q}_\beta, \dot{\leqslant}_\beta) | \beta < \kappa)$ of limit lenght κ. Let G_κ be M-generic over P_κ. Consider a sets $S \in M$, $X \in M[G_\kappa]$, $X \subseteq S$ and assume that $M[G_\kappa] \vDash \mathrm{card}(S) < \mathrm{cof}(\kappa)$. Then there is $\alpha < \kappa$ such that $X \in M[G_\alpha]$ where $G_\alpha = \{p \upharpoonright \alpha \mid p \in G_\kappa\}$.*

Proof. Take $\dot{X} \in M$ and $X = \dot{X}^{G_\kappa}$. Without loss of generality we may assume that $1_\kappa \Vdash \dot{X} \subseteq S$. Work in $M[G_\kappa]$. For all $x \in X$ choose a condition $p_x \in G_\kappa$ such that $p_x \Vdash \check{x} \in \dot{X}$. For every $x \in X$ there is some $\alpha_x < \kappa$ such that $\mathrm{supp}(p_x) \subseteq \alpha_x$. Since $\mathrm{card}(S) < \mathrm{cof}(\kappa)$ take an $\alpha < \kappa$ such that $\alpha_x \leqslant \alpha$ for all $x \in X$. We claim that
(1) $X = \{x \in S \mid \exists p \in P_\kappa (p \upharpoonright \alpha \in G_\alpha \wedge \mathrm{supp}(p) \subseteq \alpha \wedge p \Vdash \check{x} \in \dot{X})\}$.
Proof. If $x \in X$ then p_x satisfies the existential condition on the right. Conversely assume that $p \upharpoonright \alpha \in G_\alpha \wedge \mathrm{supp}(p) \subseteq \alpha \wedge p \Vdash \check{x} \in \dot{X}$. Take $q \in G_\kappa$ such that $p \upharpoonright \alpha = q \upharpoonright \alpha$. Then $\mathrm{supp}(p) \subseteq \alpha$ implies that $q \leqslant_\kappa p$. Hence $p \in G_\kappa$ and $x \in X$. *qed*(1)
This proves $X \in M[G_\alpha]$. □

Corollary 26. *In the previous lemma let P_κ have the countable chain condition and let κ be an uncountable regular cardinal. Then*

$$\mathcal{P}(\delta) \cap M[G_\kappa] = \mathcal{P}(\delta) \cap \bigcup_{\alpha < \kappa} M[G_\alpha]$$

for all $\delta < \kappa$.

7 Forcing Martin's axiom

Martin's axiom postulates the existence of partially generic sets for *all* ccc forcings. Recalling that every Cohen forcing $\mathrm{Fn}(\lambda, 2, \aleph_0)$ is ccc i.e., this amounts to a proper class of of forcings to consider. To reduce the class of requirements to a set that can be dealt with in a set-sized iterated forcing, we show that MA_κ "reflects" down to cardinality κ.

Lemma 27. *For infinite cardinals κ the following are equivalent:*

a) MA_κ*;*

b) for every ccc forcing Q whose underlying set is a subset of κ and every $\mathcal{D} \subseteq \mathcal{P}(\kappa)$ with $\mathrm{card}(\mathcal{D}) \leqslant \kappa$ *there exists a $\mathcal{D}$-*generic *filter on Q.*

Proof. $(a) \rightarrow (b)$ is obvious. For the converse use a Löwenheim-Skolem downward argument. Let $(P, \leqslant, 1)$ be a ccc forcing and let the set $\mathcal{D}$ have cardinality $\leqslant \kappa$. Without loss of generality we may assume that $\mathcal{D} \subseteq \mathcal{P}(P)$ and that every $D \in \mathcal{D}$ is dense in P. Consider the first-order structure

$$(P, \leqslant, 1, (D)_{D \in \mathcal{D}})$$

with a language of cardinality $\leqslant \kappa$. By the Löwenheim-Skolem theorem there is an elementary substructure

$$(Q, \leqslant \cap Q^2, 1, (D \cap Q)_{D \in \mathcal{D}}) \prec (P, \leqslant, 1, (D)_{D \in \mathcal{D}})$$

such that $\operatorname{card}(Q) \leqslant \kappa$. By elementarity $(Q, \leqslant \cap Q^2, 1)$ is a forcing and every $D \cap Q$ is dense in Q. If $A \subseteq Q$ is an antichain in Q then it is an antichain in P. So A is countable and Q is ccc.

We may assume that $Q \subseteq \kappa$. By (b) take a $(D \cap Q)_{D \in \mathcal{D}}$ -generic filter F on Q. We show that

$$G = \{p \in P \mid \exists q \in F q \leqslant p\}$$

is a $\mathcal{D}$-generic filter on P. The filter properties are easy. For the $\mathcal{D}$-genericity consider $D \in \mathcal{D}$. By the $(D \cap Q)_{D \in \mathcal{D}}$ -genericity of F there is $q \in F \cap (D \cap Q)$. Then

$$q \in F \cap (D \cap Q) \subseteq G \cap D \neq \emptyset .$$

$\square$

Theorem 28. *Suppose that M is a ground model. Suppose that $\kappa > \omega$ is a regular cardinal and $2^{<\kappa} = \kappa$ in M. Then there is a ccc forcing $(P_\kappa, \leq_\kappa, 1_\kappa)$ in M such that for every M-generic filter G_κ on P,* MA *and $2^\omega = \kappa$ hold in $M[G_\kappa]$.*

Proof. We shall define in M a finite support iteration $((P_\alpha, \leqslant_\alpha, 1_\alpha) | \alpha \leqslant \kappa)$ by some sequence $((\dot{Q}_\beta, \dot{\leqslant}_\beta) | \beta < \kappa)$ of names. Before choosing those names we have to make some organisatorial preparations.

(1) $\operatorname{card}(H_\kappa) = \kappa$.

Proof. Every element of H_κ can be coded as a bounded subset of κ . By $2^{<\kappa} = \kappa$ there are κ bounded subsets of κ and hence $\operatorname{card}(H_\kappa) \leqslant \kappa$. *qed.*

Fix an enumeration $H_\kappa = \{r_i | \, i < \kappa\}$.

The names $\dot{\leqslant}_\beta$ for the partial orders of the iteration will basically be the r_i in some order. Choose a bijection $h: \kappa \leftrightarrow \kappa \times \kappa \times \kappa \times \kappa$ with components $h_1, h_2, h_3, h_4: \kappa \rightarrow \kappa$ such that

$$h(\beta) = (h_1(\beta), h_2(\beta), h_3(\beta), h_4(\beta))$$

for all $\beta < \kappa$ and such that

(2) $h_1(\beta), h_2(\beta), h_3(\beta), h_4(\beta) \leqslant \beta$ for all $\beta < \kappa$.

Then

(3) for every $i, j, k < \kappa$ $\{\beta < \kappa | \, h_1(\beta) = i, h_2(\beta) = j, h_3(\beta) = k\}$ is cofinal in κ .

Let $i \leqslant \beta < \kappa$. We extend the previous operation $\pi_{i\beta}^*$ mapping P_i-names to P_β-names. Define $\pi_{i\beta}: {}^i V \rightarrow {}^\beta V$ by

$$(\pi_{i\beta}(p))(\xi) = \begin{cases} p(\xi), \text{ if } \xi < i \\ \emptyset \text{ , else} \end{cases}$$

Then define $\pi^*_{i\beta}: V \to V$ by

$$\pi^*_{i\beta}(x) = \{(\pi^*_{i\beta}(y), \pi_{i\beta}(p)) \mid (y, p) \in x \text{ and } p: i \to V\}.$$

Obviously $\pi^*_{i\beta}$ extends the previously defined embedding from P_i-names to P_β-names so that we may use the same denotations.

The sequence $((\dot{Q}_\beta, \dot{\leqslant}_\beta) | \beta < \kappa)$ is defined recursively. Suppose that $((\dot{Q}_\gamma, \dot{\leqslant}_\gamma) | \gamma < \beta)$ is already defined. This determines the finite support iteration $((P_\gamma, \leqslant_\gamma, 1_\gamma) | \gamma \leqslant \beta)$. To define $\dot{\leqslant}_\beta$ we try to obtain a fitting next partial order from the indices $i = h_1(\beta)$ and $j = h_2(\beta)$ and distinguish two cases:

Case 1. Assume that

$$1_\beta \vdash_{P_\beta} (\check{k}, \pi^*_{i\beta}(r_j), \emptyset) \text{ is a forcing satisfying the countable chain condition.}$$

Then set

$$\dot{Q}_\beta = \check{k} \text{ and } \dot{\leqslant}_\beta = \pi^*_{i\beta}(r_j).$$

Case 2. Otherwise set

$$\dot{Q}_\beta = \{\emptyset\}\check{} \text{ and } \dot{\leqslant}_\beta = \{(\emptyset, \emptyset)\}\check{}.$$

This defines the iteration $((P_\alpha, \leqslant_\alpha, 1_\alpha) | \alpha \leqslant \kappa)$.

By the iteration theorem for ccc partial orders we get that

(3) P_κ is a ccc partial order.

(4) $\mathrm{card}(P_\kappa) = \kappa$.

Proof. Conditions in P_κ have finite supports. At the elements of the support, the values of conditions are canonical names for ordinals $<\kappa$. The number of conditions is thus bounded by $\kappa^{<\omega} = \kappa$. *qed.*

Now suppose that G_κ is M-generic for P_κ. Let G_α and H_α be the derived generic filters as before. Since P_κ is ccc, cardinals are preserved between M and $M[G_\kappa]$.

(4) $M[G_\kappa] \vDash 2^\omega \leqslant \kappa$.

Proof. Counting the number of canonical names for reals we get:

$$(2^\omega)^{M[G_\kappa]} \leqslant ((\mathrm{card}(P)^\omega)^\omega)^M = (\kappa^\omega)^M = \kappa.$$

qed.

Finally we show that $M[G_\kappa] \vDash \mathrm{MA}_\lambda$ for all $\lambda < \kappa$. Let $\lambda < \kappa$. Let $(P, \leq_P, 1_P) \in M[G_\kappa]$ be a ccc partial order and let $\mathcal{D} \in M[G_\kappa]$ be a set of dense subsets of P with $|\mathcal{D}| \leq \lambda$. By previous lemmas we may assume that the domain k of P is an ordinal $<\kappa$. So $\leqslant_P \subseteq k \times k$ and the elements of $\mathcal{D}$ are subsets of k. By a previous lemma, take some $i < \kappa$ such that

$$\leqslant_P \in M[G_i] \text{ and } \mathcal{D} \subseteq M[G_i].$$

Since cardinals are preserved between $M[G_i]$ and $M[G_k]$ we have that $(P, \leq_P, 1_P)$ is a ccc partial order in $M[G_i]$. Take a P_i-name $\dot{\leq}_P \in M$ for $\leq_P$: $\dot{\leqslant}_P^{G_i} = \leqslant_P$. We may choose $\dot{\leq}_P \in H_\kappa^M$ so that

$$1_i \Vdash_{P_i} (\check{k}, \dot{\leq}_P, \emptyset) \text{ is a forcing satisfying the countable chain condition.}$$

Take $j < \kappa$ such that $r_j = \dot{\leq}_P$. Then

$$1_i \Vdash_{P_i} (\check{k}, r_j, \emptyset) \text{ is a forcing satisfying the countable chain condition.}$$

Take $\beta < \kappa$ such that $i \leqslant \beta$ and $h(\beta) = (i, j, k, *)$. Then

$$1_\beta \Vdash_{P_i} (\check{k}, \pi_{i\beta}^*(r_j), \emptyset) \text{ is a forcing satisfying the countable chain condition.}$$

By the setup of the iteration, we have

$$\dot{Q}_\beta = \check{k} \text{ and } \dot{\leqslant}_\beta = \pi_{i\beta}^*(r_j).$$

Then

$$(Q_\beta, \leqslant^{Q_\beta}, \emptyset) = (\dot{Q}_\beta^{G_\beta}, \dot{\leqslant}_\beta^{G_\beta}, \emptyset) = (k, \pi_{i\beta}^*(r_j)^{G_\beta}, \emptyset) = (k, r_j^{G_i}, \emptyset) = (k, \leqslant_P, \emptyset) = P.$$

We know that $H_\beta \in M[G_\kappa]$ is $M[G_\beta]$-generic for P. Since $\mathcal{D} \subseteq M[G_i] \subseteq M[G_\beta]$ we obtain $H_\beta \in M[G_\kappa]$ which is $\mathcal{D}$-generic for P. $\square$

8 Ideals and cardinal coefficients

Ideals capture (some aspects of) the notion of *small sets*.

Definition 29. *A set $\mathcal{I} \subseteq \mathcal{P}(R)$ is an* ideal *on R if*

a) if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$

b) if $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$

c) if $r \in R$ then $\{r\} \in \mathcal{I}$

d) $R \notin \mathcal{I}$

An ideal is κ-complete *if for any family* $\mathcal{A} \subseteq \mathcal{I}$*,* $\operatorname{card}(\mathcal{A}) < \kappa$ *holds* $\bigcup \mathcal{A} \in \mathcal{I}$*. An ideal is* σ-complete *if it is* $\aleph_1$*-complete.*

8.1 Category

Lebesgue measure defines an ideal of "small" sets, namely the ideal of measure zero sets: arbitrary subsets of measure zero sets are measure zero, and, under MA, every union of less than $2^{\aleph_0}$ measure zero sets is again measure zero.

We now look at another ideal of small sets, namely the ideal of subsets X of $\mathbb{R}$ which are nowhere dense in $\mathbb{R}$: every nonempty open interval in $\mathbb{R}$ has a nonempty open subinterval which is disjoint from X. The union of all such subintervals is open, dense in $\mathbb{R}$, and disjoint from X.

The Baire category theorem says that the intersection of countably many dense open sets of reals in dense in $\mathbb{R}$. We can strengthen this to:

Theorem 30. *Assume* MA_κ*. Then the intersection of* κ *many dense open sets of reals is dense in* $\mathbb{R}$*.*

Proof. Consider a sequence $(O_i | i < \kappa)$ of dense open subsets of $\mathbb{R}$. We use standard Cohen forcing $P = \mathrm{Fn}(\omega, 2, \aleph_0)$ for the density argument. Since P is countable it trivially has the ccc. For $i < \kappa$ define $D_i = \{p \in P | \forall x \in \mathbb{R}\, (x \supseteq p \rightarrow x \in O_i)\}$. This means that the interval determined by p lies within A_i. The density of D_i follows readily since O_i is open dense. For $n < \omega$ let $D_n = \{p \in P | n \in \mathrm{dom}(p)\}$. Obviously, D_n is also dense in P. By MA_κ let $G \subseteq P$ be $\{D_i | i < \kappa\}$-$\{D_n | n < \kappa\}$generic. Let $x = \bigcup G$. $p \in G \cap D_n$ implies that $n \in \mathrm{dom}(p) \subseteq \mathrm{dom}(x)$. So $x\colon \omega \rightarrow 2$ is a real number. □

Since $\mathrm{MA}_{\aleph_0}$ is always true in ZFC, we get the Baire category theorem:

Theorem 31. *The intersection of countably many dense open sets of reals is dense in* $\mathbb{R}$*.*

This says that dense open sets (of reals) have a largeness property, and correspondingly complements of dense open sets are small.

Definition 32. *A set* $A \subseteq \mathbb{R}$ *is* nowhere dense *if there is a dense open set* $O \subseteq \mathbb{R}$ *such that* $A \cap O = \emptyset$*. A set* $A \subseteq \mathbb{R}$ *is* meager *or* of 1st category *if it is a union of countably many nowhere dense sets.*

Proposition 33.

a) *A singleton set* $\{x\} \subseteq \mathbb{R}$ *is nowhere dense since* $\mathbb{R} \setminus \{x\}$ *is dense open in* $\mathbb{R}$*.*

b) *A countable set C is meager.*

c) *A set $A \subseteq \mathbb{R}$ is meager iff there are open dense sets $(O_n | n < \omega)$ such that $A \cap \bigcap_{n<\omega} O_n = \emptyset$.*

d) *$\mathbb{R}$ is not meager. Sets which are not meager are said to be of* 2nd category.

Proof. *c)* Let $A = \bigcup_{n<\omega} A_n$ be meager where each A_n is nowhere dense. For each n choose O_n dense open in $\mathbb{R}$ such that $A_n \cap O_n = \emptyset$. Then

$$(\bigcup_{n<\omega} A_n) \cap (\bigcap_{n<\omega} O_n) = A \cap (\bigcap_{n<\omega} O_n) = \emptyset.$$

Conversely assume that $A \cap (\bigcap_{n<\omega} O_n) = \emptyset$ where each O_n is dense open. $(A \setminus O_n) \cap O_n = \emptyset$, and so by definition, every $A_n = A \setminus O_n$ is nowhere dense. Obviously

$$\bigcup_{n<\omega} A_n \subseteq A.$$

For the converse consider $x \in A$. The property $A \cap (\bigcap_{n<\omega} O_n) = \emptyset$ implies that we may take $n < \omega$ such that $x \notin O_n$. Hence $x \in A \setminus O_n = A_n$. So $A = \bigcup_{n<\omega} A_n$ is meager.

d) If $\mathbb{R}$ were meager then there would be open dense sets $(O_n | n < \omega)$ such that $\mathbb{R} \cap \bigcap_{n<\omega} O_n = \emptyset$. But by Theorem 31,

$$\mathbb{R} \cap \bigcap_{n<\omega} O_n = \bigcap_{n<\omega} O_n \neq \emptyset,$$

contradiction. □

We would now like to show as in the case of measure that a union of $< 2^{\aleph_0}$ small sets in the sense of category is again small if Martin's axiom holds.

Theorem 34. *Assume* MA_κ *. Let $(A_i | i < \kappa)$ be a family of meager sets. Then $A = \bigcup_{i<\kappa} A_i$ is meager.*

Proof. Obviously it suffices to consider the case where each A_i is nowhere dense. We shall use MA_κ to find dense open sets $(O_n | n < \omega)$ such that

$$(\bigcup_{i<\kappa} A_i) \cap (\bigcap_{n<\omega} O_n) = A \cap (\bigcap_{n<\omega} O_n) = \emptyset.$$

The forcing will consist of approximations to a family $(O_n | n < \omega)$ of open dense sets which makes this equality true.

The forcing conditions will consist of finitely many finite approximations to the O_n. Moreover there will be for every n a finite collection of $i<\kappa$ such that an approximation to the equation holds for those i. We shall see that by appropriate density considerations the full equality may be satisfied.

For ccc-reasons, much like in the argument of measure-zero sets, we only consider approximations to the O_n by finitely many *rational* intervals. Let

$$\mathcal{I}=\{(a,b)|a,b\in\mathbb{Q},a<b\}$$

the countable set of rational open intervals $(a,b)=\{c\in\mathbb{R}|a<c<b\}$ in $\mathbb{R}$. Now let

$P=\{(r,s)|r\colon\omega\to[\mathcal{I}]^{<\omega}, s\colon\omega\to[\kappa]^{<\omega}, \{n<\omega|r(n)\neq\emptyset\}$ is finite, $\{n<\omega|s(n)\neq\emptyset\}$ is finite, $\forall n<\omega\forall i\in s(n)\ A_i\cap\bigcup r(n)=\emptyset\}$.

Define

$$(r',s')\leqslant(r,s)\ \text{ iff }\ \forall n<\omega\,(r'(n)\supseteq r(n)\wedge s'(n)\supseteq s(n)).$$

(1) $(P,\leqslant)$ satisfies the countable chain condition.
Proof. Consider (r,s) and (r,s') in P having the same first component. Then define $s''\colon\omega\to[\kappa]^{<\omega}$ by $s''(n)=s(n)\cup s'(n)$. It is easy to check that $(r,s'')\in P$, and also $(r,s'')\leqslant(r,s)$ and $(r,s'')\leqslant(r,s')$. So (r,s) and (r,s') are compatible in P.

An antichain in P must consist of conditions whose first components are pairwise distinct. Since there are only countably many first components, an antichain in P is at most countable. *qed*(1)

For each $n<\omega$ the following dense sets ensures the density of the O_n in $\mathbb{R}$: for $I\in\mathcal{I}$ let

$$D_{n,I}=\{(r',s')|\exists J\in r'(n)\ J\subseteq I\}.$$

(2) $D_{n,I}$ is dense in P.
Proof. Let $(r,s)\in P$. Let $s(n)=\{i_0,\ldots,i_{k-1}\}$. Since $A_{i_0},\ldots,A_{i_{k-1}}$ are nowhere dense one can go find intervals $I\supseteq I_{i_0}\supseteq\ldots\supseteq I_{k-1}=J$ in $\mathcal{I}$ such that $A_{i_l}\cap I_{i_l}=\emptyset$. Define $r'\colon\omega\to[\mathcal{I}]^{<\omega}$ by $r'\upharpoonright(\omega\setminus\{n\})=r\upharpoonright(\omega\setminus\{n\})$ and $r'(n)=r(n)\cup\{J\}$. Then $(r',s)\in P$, $(r',s)\leqslant(r,s)$, and $(r',s)\in D_{n,I}$. *qed*(2)

We also need that every $i<\kappa$ is considered by some O_n. Define

$$D_i=\{(r',s')|\exists n<\omega\ i\in s'(n)\}.$$

(3) D_i is dense in P.
Proof. Let $(r,s)\in P$. Take $n<\omega$ such that $r(n)=\emptyset$. Define $s'\colon\omega\to[\mathcal{I}]^{<\omega}$ by $s'\upharpoonright(\omega\setminus\{n\})=s\upharpoonright(\omega\setminus\{n\})$ and $s'(n)=s(n)\cup\{i\}$. Then $(r,s')\in P$, $(r,s')\leqslant(r,s)$, and $(r,s')\in D_i$. *qed*(3)

By MA_κ we can take a filter G on P which is generic for

$$\{D_{n,I}|n<\omega, I\in\mathcal{I}\}\cup\{D_i|i<\kappa\}.$$

For $n<\omega$ define

$$O_n=\bigcup\bigcup\{r(n)|(r,s)\in G\}.$$

(4) O_n is open, since it is a union of open intervals.
(5) O_n is dense in $\mathbb{R}$.
Proof. Let $I\in\mathcal{I}$. By genericity take $(r', s')\in G\cap D_{n,I}$. Take $J\in r'(n)$ such that $J\subseteq I$. Then

$$\emptyset\neq J\subseteq\bigcup r'(n)\subseteq\bigcup\bigcup\{r(n)|(r,s)\in G\}=O_n.$$

qed(5)
(6) Let $i<\kappa$. Then $A_i\cap\bigcap_{n<\omega}O_n=\emptyset$.
Proof. By genericity take $(r', s')\in G\cap D_i$. Take $n<\omega$ such that $i\in s'(n)$. We show that $A_i\cap O_n=\emptyset$. Assume not, and let $x\in A_i\cap O_n$. Take $(r,s)\in G$ and $I\in r(n)$ such that $x\in I$. Since G is a filter, take $(r'', s'')\in P$ such that $(r'', s'')\leqslant(r, s)$ and $(r'', s'')\leqslant(r', s')$. Then $I\in r''(n)$, $i\in s''(n)$, and

$$x\in A_i\cap I\subseteq A_i\cap\bigcup r''(n)\neq\emptyset.$$

The last inequality contradicts the definition of P. *qed*(6)

By (6), $\bigcup_{i<\kappa}A_i\cap\bigcap_{n<\omega}O_n=\emptyset$, and so $\bigcup_{i<\kappa}A_i$ is meager. $\square$

8.2 Cardinal Characteristics

We have already considered the following ideal on $\mathbb{R}$:

Definition 35. $\mathcal{N}=\{X\subseteq\mathbb{R}|X$ *has measure zero*$\}$ *is the ideal of* <u>n</u>ullsets, *the* null ideal, *and* $\mathcal{M}=\{X\subseteq\mathbb{R}|X$ *is meager*$\}$ *is the* meager ideal.

Both these ideals are σ-complete, see Theorem 10 and Theorem 34. They may have "more" completeness in certain models of set theory. We saw in the mentioned Theorem 10 that under $\mathrm{MA}_{\aleph_1}$ the ideals are $\aleph_2$-complete. On the other hand the continuum hypothesis CH implies that $\mathcal{M}$ is *not* $\aleph_2$-complete. So the value of the completeness of $\mathcal{M}$ is independent of the axioms of ZFC. To study such phenomena one introduces *cardinal characteristics* that capture properties of ideal and that may vary between different models of set theory. Sometimes these coefficients are misleadingly called cardinal *invariants*.

Definition 36. *Let $\mathcal{I}$ be an ideal on R. Define the following cardinal characteristics:*

- $\mathrm{add}(\mathcal{I}) = \min\{\mathrm{card}(\mathcal{A}) | \mathcal{A} \subseteq \mathcal{I}, \bigcup \mathcal{A} \notin \mathcal{I}\}$ *is the* additivity (number) *of $\mathcal{I}$;*
- $\mathrm{cov}(\mathcal{I}) = \min\{\mathrm{card}(\mathcal{A}) | \mathcal{A} \subseteq \mathcal{I}, \bigcup \mathcal{A} = R\}$ *is the* covering (number) *of $\mathcal{I}$;*
- $\mathrm{non}(\mathcal{I}) = \min\{\mathrm{card}(X) | X \subseteq R, X \notin \mathcal{I}\}$*;*
- $\mathrm{cof}(\mathcal{I}) = \min\{\mathrm{card}(\mathcal{A}) | \mathcal{A} \subseteq \mathcal{I}, \forall B \in \mathcal{I} \exists A \in \mathcal{A}: B \subseteq A\}$ *is the* cofinality *of $\mathcal{I}$, a family $\mathcal{A} \subseteq \mathcal{I}$ such that $\forall B \in \mathcal{I} \exists A \in \mathcal{A}: B \subseteq A$ is called* cofinal *in $\mathcal{I}$.*

Proposition 37. *Let $\mathcal{I}$ be a σ-complete ideal on $\mathbb{R}$. Then*

$$\aleph_1 \leqslant \mathrm{add}(\mathcal{I}) \leqslant \mathrm{cov}(\mathcal{I}) \leqslant \mathrm{cof}(\mathcal{I})$$

and

$$\mathrm{add}(\mathcal{I}) \leqslant \mathrm{non}(\mathcal{I}) \leqslant \mathrm{cof}(\mathcal{I})$$

This can be pictured by the following diagram:

$$\begin{tikzcd}
& & \mathrm{cov}(\mathcal{I}) \arrow[dr] & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{I}) \arrow[ur] \arrow[dr] & & \mathrm{cof}(\mathcal{I}) \\
& & \mathrm{non}(\mathcal{I}) \arrow[ur] &
\end{tikzcd}$$

Proof. The inequalities

$$\aleph_1 \leqslant \mathrm{add}(\mathcal{I}) \leqslant \mathrm{cov}(\mathcal{I}) \text{ and } \mathrm{add}(\mathcal{I}) \leqslant \mathrm{non}(\mathcal{I})$$

are trivial. To show that $\mathrm{cov}(\mathcal{I}) \leqslant \mathrm{cof}(\mathcal{I})$ consider a cofinal family $\mathcal{A} \subseteq \mathcal{I}$ with $\mathrm{card}(\mathcal{A}) = \mathrm{cof}(\mathcal{A})$. Then $\bigcup \mathcal{A} = R$ and so $\mathrm{cov}(\mathcal{I}) \leqslant \mathrm{card}(\mathcal{A}) = \mathrm{cof}(\mathcal{I})$.

To show $\mathrm{non}(\mathcal{I}) \leqslant \mathrm{cof}(\mathcal{I})$ consider again a cofinal family $\mathcal{A} \subseteq \mathcal{I}$ with $\mathrm{card}(\mathcal{A}) = \mathrm{cof}(\mathcal{A})$. For each $B \in \mathcal{A}$ choose $x_B \in \mathbb{R} \setminus B \neq \emptyset$. Then $X = \{x_B | B \in \mathcal{A}\}$ has cardinality $\leqslant \mathrm{card}(\mathcal{A}) = \mathrm{cof}(\mathcal{I})$. Assume for a contradiction that $X \in \mathcal{I}$. By cofinality take $B \in \mathcal{A}$ such that $X \subseteq B$. Then $x_B \in X \subseteq B$, contradiction. So $X \notin \mathcal{I}$ and

$$\mathrm{non}(\mathcal{I}) \leqslant \mathrm{card}(X) \leqslant \mathrm{cof}(\mathcal{I}).$$

$\square$

If the continuum hypothesis holds, then all these characteristics for the ideals $\mathcal{M}$ and $\mathcal{N}$ are equal to $\aleph_1 = 2^{\aleph_0}$. So it is interesting to study such characteristics in models of ZFC in which $\aleph_1 \neq 2^{\aleph_0}$. The obvious examples to study are models of MA + $\aleph_1 \neq 2^{\aleph_0}$ and the COHEN model for $\aleph_1 \neq 2^{\aleph_0}$.

Theorem 38. *Assume* MA. *Then*

$$\mathrm{add}(\mathcal{N}) = \mathrm{cov}(\mathcal{N}) = \mathrm{non}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = 2^{\aleph_0}$$

and

$$\mathrm{add}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = 2^{\aleph_0}$$

Proof. Because MA implies $\mathrm{add}(\mathcal{N}) = 2^{\aleph_0}$ (Theorem 10) and $\mathrm{add}(\mathcal{N}) = 2^{\aleph_0}$ (Theorem 34). $\square$

Theorem 39. *Let M be a ground model of* ZFC + CH, *and let $M \vDash \kappa$ is a regular cardinal$>\aleph_1$. In M, let $(P, \leqslant, 1_P) = \mathrm{Fn}(\omega \times \kappa, 2, \aleph_0)$ be the forcing for adding κ* COHEN *reals and let $M[G]$ be a generic extension of M by P. Then in $M[G]$*

$$\aleph_1 = \mathrm{add}(\mathcal{N}) = \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = 2^{\aleph_0}.$$

Proof. In $M[G]$, $\mathrm{cov}(\mathcal{N}) = \aleph_1$ since by Problem Sheet 1, 3(a) there is an $\aleph_1$-sequence of measure zero sets whose union is $\mathbb{R}$. $\mathrm{non}(\mathcal{N}) = 2^{\aleph_0}$, since by the argument of Theorem 2 every set of reals of cardinality $<2^{\aleph_0}$ is a measure zero set.

$\square$

Before proving an analogous result for the meager ideal $\mathcal{M}$ we make some preparations concerning "codes" of open sets in $\mathbb{R}$. In a transitive ZFC-model N consider an open set $A \subseteq \mathbb{R}$. A can be represented as

$$A = \bigcup c$$

where $c \in N$ is a set of rational open intervals. To make being a code a definite notion, only the rational endpoints of a rational interval are recorded in a code.

Definition 40. *An* open code *or a* G-code *is a set* $c \subseteq [\mathbb{Q}]^2 = \{\{r, s\} \mid r, s \in \mathbb{Q}, r < q\}$*. If* M *is a transitive model of set theory and* $c \in M$ *then*

$$c^M = \bigcup_{\{r,s\} \in c} (r, s)^M$$

is the interpretation *of the code* c *in* M*, where* $(r, s)^M = \{t \in \mathbb{R} \cap M \mid r < t < s\}$ *is the open interval between* r *and* s *as defined in* M*.*

If $N \supseteq M$ is another transitive model of set theory then $c^M \subseteq c^N$. Indeed if $\mathbb{R} \cap M \neq \mathbb{R} \cap N$ and $c \neq \emptyset$ then $c^M \neq c^N$. Nevertheless one may view c^M and c^N as the “same” open set interpreted in different models. Accordingly, many properties of c^M in M transfer to c^N in N. E.g.,

Lemma 41. *Let* $c \in M \subseteq N$ *be a G-code. Then* c^M *is dense open in* M *if* c^N *is dense open in* N*.*

Proof. Let c^M be dense open in M. Consider $r, s \in \mathbb{Q}$, $r < s$. By density take $x \in c^M \cap (r, s)^M$. Then $x \in c^N \cap (r, s)^N$.

Conversely let c^N be dense open in N. Consider $r, s \in \mathbb{Q}$, $r < s$. By density, $c^N \cap (r, s)^N \neq \emptyset$. Take a rational pair $\{r_0, s_0\} \in c$ such that $(r_0, s_0)^N \cap (r, s)^N \neq \emptyset$. Take $q \in (r_0, s_0)^N \cap (r, s)^N \cap \mathbb{Q}$. Then $q \in c^M \cap (r, s)^M$. $\square$

Note that a set $X \subseteq \mathbb{R}$ is nowhere dense iff the complement of X contains a dense open set. A set $A \subseteq \mathbb{R}$ is meager iff the complement of A contains a countable intersection of dense open sets. Let us “code” countable intersections of open sets as follows.

Definition 42. *A* G_δ-code *is a countable set* d *of G-codes. The* interpretation *of* d *is the set in a model* M *is*

$$d^M = \bigcap_{c \in d} c^M.$$

To explain the notations G and G_δ note that in Hausdorff’s times, open sets were called “Gebiet” with a “G” and countable intersections (“Durchschnitt”) were denoted by subscripts δ. We show that Cohen reals “avoid” meager sets from the ground model.

Lemma 43. *Let M be a ground model and let $M[z] = M[H]$ be a generic extension of M by the standard* COHEN *forcing $P = \mathrm{Fn}(\omega, 2, \aleph_0)$: let H be M-generic for P and let $z = \bigcup H \in {}^{\omega}2$ be the associated* COHEN *real. Consider a set $X \in M$ which is meager in the ground model and let $d \in M$ be a G_δ-code for a countable intersection of dense open sets such that $X \cap d^M = \emptyset$. Then $z \in d^{M[z]}$.*

Proof. Let us identify $\mathbb{R}$ with ${}^{\omega}2$, linearly ordered lexicographically, and let us identify $\mathbb{Q}$ with the elements of $\mathbb{R}$ which are eventually 0. Consider $c \in d$. Define, in M,

$$D = \{p \in P | \exists (r, s) \in c \forall y \in \mathbb{R} (y \supseteq p \rightarrow y \in (r, s)\}.$$

(1) D is dense in P.
Proof. Let $q \in P$. Since c^M is dense, there exists a real $y_0 \supseteq q$ such that $y_0 \in c^M$. Take $(r, s) \in c$ such that $y_0 \in (r, s)$. Take $p \in P$, $p \supseteq q$ such that $\forall y \in \mathbb{R} (y \supseteq p \rightarrow y \in (r, s))$. Then $p \in D$ and D is dense. $qed(1)$

By genericity take $p \in D \cap H$. Then $z \supseteq p$ and by the definition of D there is $(r, s) \in c$ so that

$$z \in (r, s) \subseteq c^{M[z]}.$$

Since this holds for every $c \in d$:

$$z \in \bigcap_{c \in d} c^{M[z]} = d^{M[z]}.$$

□

We can now continue to prove $\aleph_1 = \mathrm{add}(\mathcal{M}) = \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = 2^{\aleph_0}$ in the Cohen extension $M[G]$.

Lemma 44. $M[G] \vDash \mathrm{non}(\mathcal{M}) = \aleph_1$.

Proof. In $M[G]$ define the sequence $(z_i | i < \kappa)$ of COHEN reals $z_i \colon \omega \rightarrow 2$ by

$$z_i(n) = (\bigcup G)(n, i).$$

We claim that $A = \{z_i | i < \omega_1\} \notin \mathcal{M}^{M[G]}$. Assume not and let $d \in M[G]$ be a G_δ-code for a countable intersection of dense open sets so that

$$A \cap d^{M[G]} = \emptyset.$$

By previous lemmas take a countable $X \subseteq \kappa, X \in M$ such that $d \in M[G \upharpoonright X]$. Take $i \in \omega_1 \setminus X$. Then $d \in M[G \upharpoonright (\kappa \setminus \{i\})]$. We have

$$M[G] = M[G \upharpoonright (\kappa \setminus \{i\})][G \upharpoonright \{i\}] = M[G \upharpoonright (\kappa \setminus \{i\})][z_i]$$

where z_i is a COHEN real with respect to the model $M[G \upharpoonright (\kappa \setminus \{i\}]$. By the previous Lemma

$$z_i \in d^{M[G \upharpoonright (\kappa \setminus \{i\})][z_i]} = d^{M[G]}$$

contradicting that $A \cap d^{M[G]} = \emptyset$. □

Lemma 45. $M[G] \vDash \operatorname{cov}(\mathcal{M}) = 2^{\aleph_0}$.

Proof. Assume for a contradiction that $(A_\xi | \xi < \lambda)$, $\lambda < \kappa$ is a sequence of meager sets such that $\mathbb{R} = \bigcup_{\xi < \lambda} A_\xi$. For each $\xi < \lambda$ choose a G_δ-code d_ξ such that $A_\xi \cap d_\xi^{M[G]} = \emptyset$. By Lemma XXX take $X \subseteq \kappa$, $\operatorname{card}(X) = \operatorname{card}(\lambda) + \aleph_0$ such that

$$\forall \xi < \lambda \colon d_\xi \in M[G \upharpoonright X].$$

Take $i \in \kappa \setminus X$. Then

$$\forall \xi < \lambda \colon d_\xi \in M[G \upharpoonright (\kappa \setminus \{i\})].$$

As above

$$z_i \in d_\xi^{M[G \upharpoonright (\kappa \setminus \{i\})][z_i]} = d_\xi^{M[G]}$$

for all $\xi < \lambda$. Hence

$$z_i \notin \bigcup_{\xi < \lambda} A_\xi = \mathbb{R},$$

contradiction. □

9 The CICHON diagram

We want to relate cardinal characteristics of the ideals $\mathcal{N}$ and $\mathcal{M}$ in a joint diagram called the CICHON diagram. We first have to define two more characteristics.

Definition 46.

a) *Define the partial ordering* $\leqslant^*$ *of* eventual domination *on* ω^ω *by*

$$f \leqslant^* g \text{ iff } \exists m < \omega \forall n \in [m, \omega) \colon f(n) \leqslant g(n).$$

b) *The* bounding number *is*

$$\mathfrak{b} = \min \{\operatorname{card}(F) | F \subseteq {}^\omega\omega, \forall g \in {}^\omega\omega \exists f \in F \colon f \nleqslant^* g\},$$

i.e., the smallest cardinality of an unbounded family in $\leqslant^*$.

c) The dominating number *is*

$$\mathfrak{d} = \min \{\mathrm{card}(F) | F \subseteq {}^{\omega}\omega, \forall g \in {}^{\omega}\omega \, \exists f \in F \colon f \leqslant^* g\},$$

i.e., the smallest cardinality of a cofinal (or dominating) family in $\leqslant^*$.

Lemma 47. $\mathfrak{b} \leqslant \mathfrak{d}$.

Proof. Every cofinal family is unbounded. $\square$

The following diagram records provable relations between the cardinal characteristics introduced so far. An arrow $\longrightarrow$ stands for the $\leqslant$-relation between cardinals. Some inequalities have already been proved:

$$\begin{tikzcd}
\mathrm{cov}(\mathcal{N}) \arrow[r, "\text{Th. 49}"] & \mathrm{non}(\mathcal{M}) \arrow[r, "\text{L. 37}"] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \\
 & \mathfrak{b} \arrow[u] \arrow[r, "\text{L. 47}"] & \mathfrak{d} \arrow[u] & \\
\mathrm{add}(\mathcal{N}) \arrow[uu, "\text{L. 37}"'] \arrow[r] & \mathrm{add}(\mathcal{M}) \arrow[u] \arrow[r, "\text{L. 37}"'] & \mathrm{cov}(\mathcal{M}) \arrow[u] \arrow[r, "\text{Th. 49}"'] & \mathrm{non}(\mathcal{N}) \arrow[uu, "\text{L. 37}"']
\end{tikzcd}$$

It is remarkable that there are inequalities connecting the ideals $\mathcal{N}$ and $\mathcal{M}$.

Lemma 48. *There are sets $A \in \mathcal{N}$ and $B \in \mathcal{M}$ such that $A \cup B = \mathbb{R}$, i.e., $\mathbb{R}$ is the (disjoint) union of two sets which are both "small".*

Proof. We work with the standard reals $\mathbb{R}$. Let $(q_n | n < \omega)$ enumerate the rational numbers. For $m < \omega$ let

$$U_m = \bigcup_{n > m} \left(q_n - \frac{1}{2^n}, q_n + \frac{1}{2^n}\right).$$

U_m is dense open in $\mathbb{R}$ and

$$\sum_{n > m} \mathrm{length}\left(\left(q_n - \frac{1}{2^n}, q_n + \frac{1}{2^n}\right)\right) = \sum_{n > m} \frac{2}{2^n} = \frac{2}{2^m}.$$

Let $A = \bigcap_{m\in\omega} U_m$. By the calculation of the sum of interval lengths, A is a measure zero set, i.e., $A \in \mathcal{N}$.

$\mathbb{R} \setminus U_m$ is nowhere dense. Then $B = \bigcup_{m\in\omega} (\mathbb{R} \setminus U_m)$ is meager, i.e., $B \in \mathcal{M}$. Moreover

$$z \notin A \leftrightarrow \exists m < \omega: z \notin U_m \leftrightarrow \exists m < \omega: z \in (\mathbb{R} \setminus U_m) \leftrightarrow z \in B.$$

□

Theorem 49. (Rothberger, 1938) $\mathrm{cov}(\mathcal{M}) \leqslant \mathrm{non}(\mathcal{N})$ *and* $\mathrm{cov}(\mathcal{N}) \leqslant \mathrm{non}(\mathcal{M})$.

Proof. Let $A \in \mathcal{N}$ and $B \in \mathcal{M}$ such that $A \cup B = \mathbb{R}$ as in the preceding Lemma.
(1) Let $X \notin \mathcal{M}$. Then $X + A = \{x + a | x \in X, a \in A\} = \mathbb{R}$.
Proof. Let $z \in \mathbb{R}$. Then $z - X \nsubseteq B$. Take $x \in X$ such that $z - x \in A$. Then $z \in x + A \in X + A$. *qed*(1)

Now take $X \notin \mathcal{M}$ with $\mathrm{card}(X) = \mathrm{non}(\mathcal{M})$. Then

$$\mathbb{R} = X + A = \bigcup_{x\in X} (x + A).$$

The right hand side is a covering of $\mathbb{R}$ by $\leqslant \mathrm{card}(X)$ many sets in $\mathcal{N}$. So $\mathrm{cov}(\mathcal{N}) \leqslant \mathrm{card}(X) = \mathrm{non}(\mathcal{M})$.

The proof of the other inequality proceeds in the same way, with $\mathcal{M}$ and $\mathcal{N}$ interchanged. □

Before we prove further inequalities in the Cichon diagram let us check the values in the diagram in the models of set theory considered so far.

If we assume MA or CH then we know already that all entries except possible $\mathfrak{b}$ or $\mathfrak{d}$ are equal to $2^{\aleph_0}$.

Lemma 50. *Assume* MA. *Then* $\mathfrak{b} = 2^{\aleph_0}$ *(and so* $\mathfrak{d} = 2^{\aleph_0}$*).*

Proof. Let $F \subseteq {}^{\omega}\omega$ and $\mathrm{card}(F) < 2^{\aleph_0}$. It suffices to show that F is bounded in the structure $({}^{\omega}\omega, \leqslant^*)$. Define Hechler *forcing* by

$$P = \{(a, A) | a \in {}^{<\omega}\omega, A \subseteq {}^{\omega}\omega, \mathrm{card}(A) < \aleph_0\}$$

with

$$(a', A') \leqslant (a, A) \text{ iff } a' \supseteq a, A' \supseteq A, \text{ and } \forall n \in \mathrm{dom}(a') \setminus \mathrm{dom}(a)\, \forall f \in A: a'(n) > f(n)$$

and $1_P = (\emptyset, \emptyset)$.
(1) HECHLER forcing has the ccc.
Proof. If $(a, A), (a, B) \in P$ with the same "stem" a, then they are compatible:

$$(a, A \cup B) \leqslant (a, A), (a, B).$$

So if $\mathcal{C}$ is an antichain in P, then the map $(a,\ A) \mapsto a$ is injective on $\mathcal{C}$. Since there are only countably many possible stems a, $\mathrm{card}(\mathcal{C}) \leqslant \aleph_0$. *qed*(1)

For every $f \in {}^\omega\omega$ set

$$D_f = \{(a, A) \in P | f \in A\}.$$

(2) D_f is dense in P.
Proof. Since $(a, A \cup \{f\}) \leqslant (a, A)$ and $(a, A \cup \{f\}) \in D_f$. *qed*(2)

For every $n < \omega$ set

$$D_n = \{(a, A) \in P | n \in \mathrm{dom}(a)\}.$$

(3) D_n is dense in P.
Proof. Let $(b, B) \in P$. Define $a: n+1 \to \omega$ by

$$a(i) = \begin{cases} b(i), \text{ if } i \in \mathrm{dom}(b) \\ \max\{f(i) | f \in B\} + 1 \end{cases}$$

Then $(a, B) \leqslant (b, B)$ and $(a, A) \in D_n$. *qed*(3)

By MA take a $\{D_f | f \in F\} \cup \{D_n\}$-generic filter G on P. Let

$$h = \bigcup \{a | (a, A) \in G\}.$$

Then $h: \omega \to \omega$, since G meets every D_n.
(4) $\forall f \in F: f \leqslant^* h$, i.e., F is bounded.
Proof. Let $f \in F$. Take $(a, A) \in G \cap D_f$. Let $m = \mathrm{dom}(a)$. Consider $n \in [m, \omega)$. Let $(a', A') \in G$ such that $n \in \mathrm{dom}(a')$. Since all elements of G are compatible we may assume that $(a', A') \leqslant (a, A)$. Then

$$h(n) = a'(n) > f(n).$$

Hence $h \geqslant^* f$. $\square$

So under MA or CH all entries in the CICHON diagram are equal to $2^{\aleph_0}$.

In the COHEN model for $2^{\aleph_0} = \kappa > \aleph_1$ we have from our previous analysis:

$$\begin{tikzcd}
\operatorname{cov}(\mathcal{N}) = \aleph_1 \arrow[r] & \operatorname{non}(\mathcal{M}) = \aleph_1 \arrow[r] & \operatorname{cof}(\mathcal{M}) = 2^{\aleph_0} \arrow[r] & \operatorname{cof}(\mathcal{N}) = 2^{\aleph_0} \\
 & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & \\
\operatorname{add}(\mathcal{N}) = \aleph_1 \arrow[r] \arrow[uu] & \operatorname{add}(\mathcal{M}) = \aleph_1 \arrow[r] \arrow[u] & \operatorname{cov}(\mathcal{M}) = 2^{\aleph_0} \arrow[r] \arrow[u] & \operatorname{non}(\mathcal{N}) = 2^{\aleph_0} \arrow[uu]
\end{tikzcd}$$

We now determine that the values of $\mathfrak{b}$ and $\mathfrak{d}$ are consistent with the diagram:

Theorem 51. *Let M be a ground model of* ZFC + CH, *and let $M \vDash \kappa$ is a regular cardinal$>\aleph_1$. Let $M[G]$ be a generic extension of M by the partial order for adjoining κ* Cohen *reals using finite conditions. Then, in $M[G]$, $\mathfrak{b} = \aleph_1$ and $\mathfrak{d} = 2^{\aleph_0}$.*

Proof. We show that the first $\aleph_1$ Cohen reals are unbounded. On the other hand no family $<2^{\aleph_0}$ can be cofinal in ${}^{\omega}\omega$ since there will always be a Cohen real which is not dominated. □

10 $\operatorname{cov}(\mathcal{M}) \leqslant \mathfrak{d}$

To give an impression of non-trivial proofs of inequalities in Cichon's diagram we show that $\operatorname{cov}(\mathcal{M}) \leqslant \mathfrak{d}$. Recall that

$$\mathfrak{d} = \min \{\operatorname{card}(F) | F \subseteq {}^{\omega}\omega, \forall g \in {}^{\omega}\omega \exists f \in F : f \leqslant^* g\},$$

is the smallest cardinality of a cofinal (or dominating) family in $\leqslant^*$.

It is convenient to introduce the following quantifiers:

$$\exists^{\infty} n \varphi(n) \quad \text{for} \quad \forall m \in \omega \exists n \in \omega (n > m \wedge \varphi(n)) \text{ “there are infinitely many”}$$

$$\forall^{\infty} n \varphi(n) \quad \text{for} \quad \exists m \in \omega \forall n \in \omega (n > m \rightarrow \varphi(n)) \text{ “for all but finitely many”.}$$

The following theorem links the meager ideal to a combinatorial property in ${}^{\omega}\omega$:

Theorem 52. $\operatorname{cov}(\mathcal{M}) = \min \{\operatorname{card}(F) |\, F \subseteq {}^{\omega}\omega \textit{and } \forall g \in {}^{\omega}\omega \exists f \in F \forall^{\infty} n\, f(n) \neq g(n)\}$.

The family F on the RHS can be considered to be "cofinal" for the relation $\forall^\infty n\ f(n) \neq g(n)$ of being *eventually different*. Since $f <^* g$ implies $\forall^\infty n\ f(n) \neq g(n)$ the Theorem implies the desired inequality.

$$\begin{aligned}\mathrm{cov}(\mathcal{M}) &= \min\{\mathrm{card}(F)|\ F\subseteq {}^\omega\omega \text{and } \forall g\in {}^\omega\omega \exists f\in F \forall^\infty n\, f(n)\neq g(n)\}\\ &\leqslant \min\{\mathrm{card}(F)| F\subseteq {}^\omega\omega, \forall g\in {}^\omega\omega\exists f\in F\colon f\leqslant^* g\}\\ &= \mathfrak{d}\end{aligned}$$

Theorem 65 will follow from the following

Lemma 53. *For every infinite cardinal κ the following are equivalent:*

1. *$\mathbb{R}$ is not the union of less than κ-many meager sets;*
2. $\forall F\in [{}^\omega\omega]^{<\kappa} \exists g\in {}^\omega\omega \forall f\in F \exists^\infty n\, f(n) = g(n)$*;*
3. $\forall F\in [{}^\omega\omega]^{<\kappa}\,\forall G\in [[\omega]^\omega]^{<\kappa}\,\exists g\in {}^\omega\omega \forall f\in F\forall X\in G\exists^\infty n\in X\, f(n)=g(n)$*.*

Note that $a)$ expresses that $\kappa \leqslant \mathrm{cov}(\mathcal{M})$; $b)$ expresses that $\kappa \leqslant$ the RHS in Theorem 52. This implies the equality in Theorem 52.

Proof. $(1)\to(2)$: Assume $F\subseteq {}^\omega\omega$ and $|F|<\kappa$. For $f\in F$ let $G_f = \{g\in {}^\omega\omega|\ \exists^\infty n\ f(n) = g(n)\}$. Every $G_f = \bigcap_{m\in\omega}\{g\in {}^\omega\omega|\ \exists n>m\ f(n)=g(n)\}$ is a G_δ-set because every $\{g\in {}^\omega\omega|\ \exists n>m\ f(n)=g(n)\}$ is open. Moreover, every G_f is dense in $\mathbb{R}$. Hence $\mathbb{R}-G_f$ is meager for all $f\in F$. So

$$\bigcup_{f\in F}(\mathbb{R}-G_f)\neq\mathbb{R}.$$

Take

$$g\in\mathbb{R}\setminus\bigcup_{f\in F}(\mathbb{R}-G_f)=\bigcap\{G_f|\ f\in F\}.$$

Then $\forall f\in F\exists^\infty n\, f(n)=g(n)$, as required.

$(2)\to(3)$: Let $F=\{f_\alpha|\ \alpha<\lambda\}$, $\lambda<\kappa$, be a family of functions $f_\alpha\colon\omega\to\omega$ and $G=\{X_\alpha|\ \alpha<\lambda\}$ be a family of infinite subsets of ω. Let $\langle x_\alpha^n|\ n\in\omega\rangle$ be the monotone enumeration of X_α. Let $Q=\{s\colon \mathrm{dom}(s)\to\omega|\ \mathrm{dom}(s)\subseteq\omega \text{ is finite}\}$ the set of all finite partial functions from ω to ω; this is a version of Cohen forcing for a single Cohen real. For $\alpha,\beta<\lambda$ define a function $h_{\alpha,\beta}\colon\omega\to Q$ by

$$h_{\alpha,\beta}(n)=f_\beta\restriction\{x_\alpha^0,x_\alpha^1,\dots,x_\alpha^n\}$$

for all $n \in \omega$. Since Q is countable, we have by (2)

$$\forall F \in [{}^{\omega}Q]^{<\kappa} \exists h \in {}^{\omega}Q \forall f \in F \exists^{\infty} n\, f(n) = h(n).$$

In particular, there exists a function $h \in {}^{\omega}Q$, such that

$$\forall \alpha, \beta < \lambda \exists^{\infty} n\, h_{\alpha,\beta}(n) = h(n).$$

Recursively choose a sequence $\langle x_n | \, n \in \omega \rangle$, such that

$$x_n \in \mathrm{dom}(h(n)) - \{x_0, x_1, \ldots, x_{n-1}\}$$

for all $n \in \omega$. Let $g: \omega \to \omega$ be a function, such that for all $n < \omega$

$$g(x_n) = h(n)(x_n).$$

To check (3) for g consider $\alpha, \beta < \lambda$. There are infinitely many n with $h_{\alpha,\beta}(n) = h(n)$. For such n the corresponding x_n satisfies

$$x_n \in \mathrm{dom}(h(n)) = \mathrm{dom}(h_{\alpha,\beta}(n)) = \{x_\alpha^0, x_\alpha^1, \ldots, x_\alpha^n\} \subseteq X_\alpha$$

and

$$f_\beta(x_n) = h_{\alpha,\beta}(n)(x_n) = h(n)(x_n) = g(x_n).$$

Thus $\exists^{\infty} n \in X_\alpha\, f_\beta(n) = g(n)$

(3) $\to$ (1): □

Proof:
(1) $\to$ (2): Assume $F \subseteq \omega^\omega$ and $|F| < \kappa$. For $f \in F$ let $G_f = \{g \in \omega^\omega | \, \exists^{\infty} n\ f(n) = g(n)\}$. Every $G_f = \bigcap_{m \in \omega} \{g \in \omega^\omega | \, \exists n > m\ f(n) = g(n)\}$ is a G_δ-set because every $\{g \in \omega^\omega | \, \exists n > m\, f(n) = g(n)\}$ is open. Moreover, every G_f is dense in $\langle \mathsf{R} \rangle$. Hence $\langle \mathsf{R} \rangle - G_f$ is for all $f \in F$ meager. So $\bigcap \{G_f | \, f \in F\} \neq \emptyset$. But if $g \in \bigcap \{G_f | \, f \in F\}$, then $\forall f \in F \exists^{\infty} n\ f(n) = g(n)$.
(2) $\to$ (3): Let $F = \{f_\alpha | \, \alpha < \lambda\}$, $\lambda < \kappa$, be a family of functions $f_\alpha: \omega \to \omega$ and $G = \{X_\alpha | \, \alpha < \lambda\}$ be a family of infinite subsets of ω. Let $\langle x_\alpha^n | \, n \in \omega \rangle$ be the monotone enumeration of X_α. For $\alpha, \beta < \lambda$ define a function $h_{\alpha,\beta}$ by

$$h_{\alpha,\beta} = f_\beta \restriction \{x_\alpha^0, x_\alpha^1, \ldots, x_\alpha^n\}$$

for all $n \in \omega$. Let $\Phi = \{s: d\,o\,m(s) \to \omega | \, d\,o\,m(s) \subseteq \omega \text{ finite}\}$. Since Φ is countable, we have by (2)

$$\forall F \in [\Phi^\omega]^{<\kappa} \exists h \in \Phi^\omega \forall f \in F \exists^{\infty} n\, f(n) = h(n).$$

In particular, there exists a function $h \in \Phi^\omega$, such that

$$\forall \alpha, \beta < \lambda \exists^\infty n\, h_{\alpha,\beta}(n) = h(n).$$

Pick inductively a sequence $\langle x_n | \, n \in \omega \rangle$, such that

$$x_n \in dom(h(n)) - \{x_0, x_1, \ldots, x_{n-1}\}$$

for all $n \in \omega$. Let $g\colon \omega \to \omega$ be a function, such that

$$g(x_n) = h(n)(x_n)$$

holds for all n with $h_{\alpha,\beta}(n) = h(n)$. Then g is a witness for (3) because

$$f_\beta(x_n) = h_{\alpha,\beta}(n)(x_n) = h(n)(x_n) = g(x_n)$$

for all n with $h_{\alpha,\beta}(n) = h(n)$.
(3) $\to$ (1): Let $\langle F_\alpha | \, \alpha < \lambda \rangle$ with $\lambda < \kappa$ be a family of meager sets. We want to show that $\bigcup \{F_\alpha | \; \alpha < \lambda\} \neq \langle \mathsf{R} \rangle$. Since every F_α is meager, $F_\alpha = \bigcup \{F_\alpha^n | \; n \in \omega\}$ where every F_α^n is nowhere dense. By definition also the topological closure $cl(F_\alpha^n)$ is nowhere dense. So we can assume w.l.o.g. that $\langle F_\alpha | \, \alpha < \lambda \rangle$ is a family of closed nowhere dense sets.
For $\alpha < \lambda$ let

$$s_n^\alpha = min\{s \in 2^{<\omega} | \, \forall t \in 2^{<n} [t^\frown s] \cap F_\alpha = \emptyset\}$$

where the minimum is taken with respect to a fixed enumeration of $2^{<\omega}$ and $[s] = \{f \in 2^\omega | \; s \subseteq f\}$. Why does this minimum exist? Consider first an arbitrary $t \in 2^{<\omega}$. Then there exists $t \subseteq s \in 2^{<\omega}$, such that

$$[s] \cap F_\alpha = \emptyset. \quad (*)$$

Because: $[t]$ is open. Since F_α is nowhere dense, $[t] \nsubseteq F_\alpha$. Pick $f \in [t] \setminus F_\alpha$. But $2^\omega - F_\alpha$ is open. So there exists a neighbourhood $[s]$ of f such that $[s] \subseteq 2^\omega - F_\alpha$.
Now we can construct recursively an $s \in 2^{<\omega}$ such that

$$\forall t \in 2^{<n} [t^\frown s] \cap F_\alpha = \emptyset.$$

To do so, let $2^{<n} = \{t_k | \; k \leq m\}$. For t_0 pick an s_0 as in $(*)$. If s_k is already defined, consider $t_{k+1}^\frown s_k$ and pick for it an s_{k+1} as in $(*)$. Then s_m is as wanted.
Back to the s_n^α from above. By (3) there exists a sequence $\langle s_n | \, n \in \omega \rangle$ such that

$$\forall \alpha < \lambda \exists^\infty n\, s_n^\alpha = s_n.$$

For $\alpha<\lambda$ let $X_\alpha=\{n\in\omega|\, s_n^\alpha=s_n\}$.
Lemma
There exists an increasing sequence $\langle k_n|\, n\in\omega\rangle$ such that
(1) $\sum_{j\le k_n}|s_j|<k_{n+1}$ for all $n\in\omega$
(2) $\forall\alpha<\lambda\exists^\infty n\, x_\alpha\cap[k_{2n},k_{2n+1}[\neq\emptyset$.
Proof: For every finite $A\subseteq\lambda$ define $f_A\colon\omega\to\omega$ by

$$f_A(n)=m\,i\,n\{m\in\omega|\,\forall\alpha\in A[n,m[\cap X_\alpha\neq\emptyset\}$$

and for every $k\in\omega$ let

$$f'_{A,k}(0)=k \quad\text{and}\quad f'_{A,k}(n+1)=f_A(f'_{A,k}(n))$$

for all $n\in\omega$.
By (3) $\lambda<\mathfrak{d}$. So there exists a strictly increasing function $f\colon\omega\to\omega$ such that
1. $\forall A\in[\lambda]^{<\omega}\forall k\exists^\infty n\, f'_{A,k}(n)<f(n)$
2. $\sum_{j\le f(n)}|s_j|<f(n+1)$.
We can find such an f because $|\{f'_{A,k}|\, A\in[A]^{<\omega},k\in\omega\}|=|\lambda|$. So there is by definition of $\mathfrak{d}$ an f which is not dominated by any $f'_{A,k}$. That is $f\not\le^* f'_{A,k}$ for all $A\in[\lambda]^{<\omega}$, $k\in\omega$, i.e. $\exists^\infty n\, f'_{A,k}(n)<f(n)$. Once we have found such an f we can recursively ensure 2.
We have

$$\forall A\in[\lambda]^{<\omega}\exists^\infty n\exists k\, f(n)\le k\le f_A(k)\le f(n+1)\quad(*).$$

Otherwise there were A and m such that

$$\forall n\ge m\, f(n+1)<f_A(f(n)).$$

So for $k=f(m)$ and all $m\in\omega$

$$f(n)\le f(n+m)<f_A(f(m))<f_A(f_A(f(m)))<\ldots<f'_{A,k}(n)$$

would hold. But this contradicts the choice of f.
Define $X'_A=\{n\in\omega|\,\exists k\, f(n)\le k\le f_A(k)<f(n+1)\}$. Then by $(*)$ X'_A is infinite for all $A\in[\lambda]^{<\omega}$. Consider $X^0=\{2n|\, n\in\omega\}$ and $X^1=\{2n+1|\, n\in\omega\}$. Then ($X'_A\cap X^0$ is infinite for all $A\in[\lambda]^{<\omega}$) or ($X'_A\cap X^1$ is infinite for all $A\in[\lambda]^{<\omega}$). [If otherwise $|X'_A\cap X^0|<\omega$ and $|X'_B\cap X^1|<\omega$, then $|X'_A\cap X'_B|=|X'_{A\cup B}|=|X'_{A\cup B}\cap X^0|+|X'_{A\cup B}\cap X^1|\le|X'_A\cap X^0|+|X'_B\cap X^1|<\omega$ which contradicts $(*)$.]
If $|X'_A\cap X^0|=\omega$, then set $k_n=f(n)$. Otherwise set $k_n=f(n+1)$.
$\square$ (Lemma)
Lemma
There exists $X\subseteq\omega$ such that $|X\cap[k_{2n},k_{2n+1}[|\le 1$ for all $n\in\omega$ and $X\cap X_\alpha$ is infinite for every $\alpha<\lambda$.
Proof: For $\alpha<\lambda$ and $n\in\omega$ define

$f_\alpha(n) = min(X_\alpha \cap [k_{2n}, k_{2n+1}[)$ if $X_\alpha \cap [k_{2n}, k_{2n+1}[\neq \emptyset$

$f_\alpha(n) = 0$ otherwise
and $Y_\alpha = \{n \in \omega | f_\alpha(n) \neq \emptyset\}$. By (3) there exists a $g \in \omega^\omega$ such that

$$\forall \alpha < \lambda \exists^\infty n \in Y_\alpha g(n) = f_\alpha(n).$$

Hence the claim holds for $X = \{g(n) | n \in \omega\}$. $\square$
Now we can prove "(3) → (1)". Let $\langle x_n | n \in \omega\rangle$ be the monotone enumeration of X from the previous lemma. Set

$$x = s_{x_0}{}^\frown s_{x_1}{}^\frown s_{x_2}{}^\frown \ldots$$

We show that $x \notin \bigcup \{F_\alpha | \alpha < \lambda\}$. Let $\alpha < \lambda$. It follows from the above construction that there exists $x_n \in X \cap [k_{2n}, k_{2n+1}[$ such that $s_{x_n} = s^\alpha_{x_n}$. But $\sum_{j<n} |s_{x_j}| < k_{2n} < x_n$ and (by the definition of $\langle s^\alpha_n | n \in \omega\rangle$) $[s_{x_0}{}^\frown \ldots {}^\frown s_{x_{n-1}}{}^\frown s_{x_n}] = [s_{x_0}{}^\frown \ldots {}^\frown s_{x_{n-1}}{}^\frown s^\alpha_{x_n}]$ is disjoint from F_α. Hence $x \notin F_\alpha$. $\square$

11 Proper Forcing

Definition 54. $H_\lambda = \{x \mid \mathrm{card}(\mathrm{TC}(x)) < \lambda\}$. *We assume that every H_λ has a chosen wellorder $<$.*

Definition 55. $(M, \in, <) \prec (H_\lambda, \in, <)$ *iff for every $\varphi \in \mathrm{Fml}(\in, <)$ and every $\vec{a} \in \mathrm{Asn}(M)$*

$$(M, \in, <) \models \varphi[\vec{a}] \text{ iff } (H_\lambda, \in, <) \models \varphi[\vec{a}].$$

We simply write $M \prec H_\lambda$ instead of $(M, \in, <) \prec (H_\lambda, \in, <)$.

Definition 56. *Let $M \prec H_\lambda$ and let $(P, \leqslant) \in M$ be a forcing. Let G be V-generic on P. Then define*

$$M[G] = \{x^G \mid x \in M\}.$$

This definition will relate to the notions of a generic condition and properness.

Lemma 57. *Let $M \prec H_\lambda$ and let $(P, \leqslant) \in M$ be a forcing. Let G be V-generic on P. Then $H_\lambda[G] = H_\lambda^{V[G]}$ and*

$$M[G] \prec H_\lambda[G].$$

Proof. Let $x \in H_\lambda[G]$. Let $\dot{x} \in H_\lambda$ and $\dot{x}^G = x$. By the definition of the interpretation function

$$\mathrm{TC}(x) \subseteq \{\dot{y}^G \mid \dot{y} \in \mathrm{TC}(\dot{x})\}$$

Hence

$$V[G] \vDash \mathrm{card}(\mathrm{TC}(x)) \leqslant \mathrm{card}(\mathrm{TC}(\dot{x})) < \lambda$$

and $x \in H_\lambda^{V[G]}$.

Conversely, let $x \in H_\lambda^{V[G]}$. ...

□

Definition 58. *Let $M \prec H_\lambda$ and let $(P, \leqslant) \in M$ be a forcing. $q \in P$ is (M, P)-*generic *iff for every $D \in M$ which is dense in P, $D \cap M$ is* predense *below q, i.e.,*

$$\forall q_1 \leqslant q \exists q_2 \leqslant q_1 \exists d \in D \cap M q_2 \leqslant d.$$

Lemma 59. *$q \in P$ is (M, P)-*generic *iff for every $D \in M$ which is dense in P there is a P-name $\dot{p}$ such that*

$$q \Vdash \dot{p} \in D \cap M \cap \dot{G}.$$

Proof. Let $q \in P$ be (M, P)-generic and let $D \in M$ be dense in P. Let G be V-generic on P with $q \in G$. By the definition of being (M, P)-generic the set

$$\{q_2 \mid \exists d \in D \cap M q_2 \leqslant d\}$$

is dense below q. By the genericity of G take $q_2 \in G$ such that $\exists d \in D \cap M q_2 \leqslant d$. Take $p \in D \cap M$ such that $q_2 \leqslant p$. Then $p \in D \cap M \cap G$. Thus

$$q \Vdash \exists p p \in D \cap M \cap G.$$

By the maximality principle there is a P-name $\dot{p}$ such that

$$q \Vdash \dot{p} \in D \cap M \cap \dot{G}.$$

For the converse, assume the RHS of the equivalence. To show that q is (M, P)-generic consider $D \in M$ which is dense in P. Let $\dot{p}$ be a P-name such that

$$q \Vdash \dot{p} \in D \cap M \cap \dot{G}.$$

To show that $D\cap M$ is predense below q let $q_1\leqslant q$. $q_1\Vdash\dot{p}\in D\cap M\cap\dot{G}$. Take a condition $q_2\leqslant q_1$ and a $d\in D\cap M$ such that

$$q_2\Vdash\dot{p}=\check{d}\wedge\check{d}\in\dot{G}\,.$$

Then q_2 and d must be compatible in P. Take $q_3\leqslant q_2, d$. q_3 and d witness the predensity of $D\cap M$. □

Lemma 60. *A condition $q\in P$ is (M,P)-generic iff*

$$q\Vdash M[\dot{G}]\cap\mathrm{Ord}=M\cap\mathrm{Ord}\,.$$

Proof. Let $q\in P$ be (M, P)-generic. Let G be V-generic on P with $q\in G$. Let $\alpha\in M[G]\cap\mathrm{Ord}$. Take a P-name $\dot{\alpha}\in M$ such that $\alpha=\dot{\alpha}^G$. We may assume that $1_P\Vdash\dot{\alpha}\in\mathrm{Ord}$. The set

$$D=\{d\in P\mid\exists\beta\in\mathrm{Ord}\,d\Vdash\dot{\alpha}=\check{\beta}\}\in M$$

is dense in P. By assumption, $D\cap M$ is predense below $q\in G$. So there is $d\in D\cap M\cap G$.

$$H_\lambda\vDash\exists\beta\in\mathrm{Ord}\,d\Vdash\dot{\alpha}=\check{\beta}\,.$$

Since $M\prec H_\lambda$

$$M\vDash\exists\beta\in\mathrm{Ord}\,d\Vdash\dot{\alpha}=\check{\beta}\,.$$

Take $\beta\in M\cap\mathrm{Ord}$ such that $d\Vdash\dot{\alpha}=\check{\beta}$. Then $\alpha=\dot{\alpha}^G=\beta\in M$.

Conversely let $q\in P$ *not* be (M, P)-generic. Take a dense set $D\in M$ such that $D\cap M$ is *not* predense below q. Let $A\subseteq D$ be a maximal antichain with $A\in M$. Define a P-name for an ordinal by

$$\dot{\alpha}=\{(\check{\beta},a)\mid a\in A\text{ is the }\beta\text{-th element of }H_\lambda\text{ in the chosen wellorder of }H_\lambda\}\in M.$$

Since $D\cap M$ is not predence below q take $q_1\leqslant q$ which is incompatible with every element of $D\cap M$. Let G be V-generic with $q_1\leqslant q$. Let $\alpha=\dot{\alpha}^G$. This is due to the fact that there is $a\in A\cap G$ such that a is the α-th element of H_λ. Assume for a contradiction that $\alpha\in M\cap\mathrm{Ord}$. Then $a\in A\cap M\cap G\subseteq D\cap M\cap G$. But then q_1 is compatible with $a\in D\cap M$, contradiction. Thus

$$M[G]\cap\mathrm{Ord}\neq M\cap\mathrm{Ord}.$$

□

Definition 61. *A forcing* $(P, \leqslant)$ *is* proper *iff for every* $\lambda > 2^{\mathrm{card}(P)}$ *and every countable* $M \prec H_\lambda$ *with* $P \in M$ *and every* $p \in P \cap M$ *there is* $q \leqslant p$ *which is* (M, P)*-generic.*

Lemma 62. $(P, \leqslant)$ *is proper iff for every* $\lambda > 2^{\mathrm{card}(P)}$ *and every countable* $M \prec H_\lambda$ *with* $P \in M$ *and every* $p \in P \cap M$ *there is* $q \leqslant p$ *such that for every* V*-generic* G *with* $q \in G$

$$M[G] \cap \mathrm{Ord} = M \cap \mathrm{Ord}.$$

Theorem 63. *Let* $V[G]$ *be a generic extension by a proper forcing* $(P, \leqslant)$*. Then*

a) *for every* $a \in ([\mathrm{Ord}]^\omega)^{V[G]}$ *there is* $b \in ([\mathrm{Ord}]^\omega)^V$ *such that* $a \subseteq b$*;*

b) $\aleph_1^{V[G]} = \aleph_1^V$.

Proof. (a) Let $a \in ([\mathrm{Ord}]^\omega)^{V[G]}$ and take $\dot{f} \in V$, $\dot{f}^G: \omega \to a$. Take $p \in G$ such that $p \Vdash \dot{f}: \omega \to \mathrm{Ord}$. Take $\lambda \in \mathrm{Card}$ sufficiently high with $p, P, \dot{f} \in H_\lambda$. Since $(P, \leqslant)$ is proper the set

$$D = \{q \in P \mid \text{there is a countable } M \prec H_\lambda \text{ with } p, P, \dot{f} \in M \text{ and } q \leqslant p \text{ is } (M, P)\text{-generic}\}$$

is dense in P below p. By the genericity of G take $q \in D \cap G$ and a countable $M \prec H_\lambda$ such that $p, P, \dot{f} \in M$ and $q \leqslant p$ is (M, P)-generic. By Lemma 98

$$M[G] \cap \mathrm{Ord} = M \cap \mathrm{Ord}.$$

Set $b = M \cap \mathrm{Ord} \in ([\mathrm{Ord}]^\omega)^V$.

(1) $a \subseteq b$.

Proof. Let $x \in a$. Let $x = \dot{f}^G(n)$. By the maximality principle there is a canonical name $\dot{x} \in H_\lambda$ such that $p \Vdash \dot{x} = \dot{f}(\check{n})$. Since $M \prec H_\lambda$ we may assume $\dot{x} \in M$. Then

$$x = \dot{x}^G \in M[G] \cap \mathrm{Ord} = M \cap \mathrm{Ord} = b$$

(b) follows immediately from (a). $\square$

Many important forcings are proper:

Lemma 64. *If* $(P, \leqslant)$ *is ccc then it is proper.*

Proof. Let $\lambda > 2^{\mathrm{card}(P)}$, $M \prec H_\lambda$ countable with $P \in M$, and $p \in P \cap M$. We show that p itself is an (M, P)-generic condition. Let G be V-generic for P with $p \in G$. It suffices to show that $M[G] \cap \mathrm{Ord} = M \cap \mathrm{Ord}$. Let $\alpha \in M[G] \cap \mathrm{Ord}$. Take $\dot{\alpha} \in M$ such that $\alpha = \dot{\alpha}^G$ and $\Vdash \dot{\alpha} \in \mathrm{Ord}$. Let

$$A = \{\beta \in \mathrm{Ord} \mid \exists r \leqslant p . r \Vdash \dot{\alpha} = \check{\beta}\}$$

be a set of possible interpretations of $\dot{\alpha}$. We can define a function $A \to P$, $\beta \mapsto r_\beta$ such that $r_\beta \Vdash \dot{\alpha} = \check{\beta}$. $\{r_\beta \mid \beta \in A\}$ is an antichain in P. By the ccc, $\{r_\beta \mid \beta \in A\}$ is at most countable and so A is at most countable. $A \in M$ and $M \vDash A$ is countable. So $A \subseteq M$. Thus

$$\alpha \in A \subseteq M.$$

□

Lemma 65. *If $(P, \leqslant)$ is countably complete then it is proper.*

Proof. Let $\lambda > 2^{\mathrm{card}(P)}$, $M \prec H_\lambda$ countable with $P \in M$, and $p \in P \cap M$. Let $(x_n \mid n < \omega)$ be an enumeration of M. Define sequences $(p_n \mid n < \omega) \subseteq P \cap M$ and $(\alpha_n \mid n < \omega) \subseteq \mathrm{Ord}$ such that

$$p \geqslant p_0 \geqslant p_1 \geqslant \ldots$$

Choose a condition $p_0 \leqslant p$, $p_0 \in M$ and $\alpha_0 \in \mathrm{Ord}$ such that $p_0 \Vdash x_0 = \check{\alpha}_0$ if that is possible; otherwise let $p_0 = p$ and $a_0 = 0$. If $p_n \in P \cap M$ is defined, choose a condition $p_{n+1} \leqslant p_n$, $p_{n+1} \in M$ and $\alpha_{n+1} \in \mathrm{Ord}$ such that $p_{n+1} \Vdash x_{n+1} = \check{\alpha}_{n+1}$ if that is possible; otherwise let $p_{n+1} = p_n$ and $a_{n+1} = 0$. Note that $(\alpha_n \mid n < \omega) \subseteq M$ since every α_n is definable from p_n, $x_n \in M$.

By the countable completeness of P take $q \in P$ such that $\forall n < \omega q \leqslant p_n$. We show that q is an (M, P)-generic condition. Let G be V-generic for P with $q \in G$. It suffices to show that $M[G] \cap \mathrm{Ord} = M \cap \mathrm{Ord}$. Let $\alpha \in M[G] \cap \mathrm{Ord}$. Take $x_n \in M$ such that $\alpha = x_n^G$ and some $r \in G$ such that $r \Vdash x_n = \check{\alpha}$. By the definition of $(p_n \mid n < \omega)$ this means that $p_n \Vdash x_n = \check{\alpha}_n$. Since $r, p_n \in G$ are compatible

$$\alpha = \alpha_n \in M.$$

□

12 2-step iterations of proper forcing

Lemma 66. *Let P be a forcing and $\Vdash_P \dot{Q}$ is a forcing. Let $M \prec H_\lambda$ and $P * \dot{Q} \in M$. Then $(q_0, \dot{q}_1)$ is $(M, P * \dot{Q})$-generic iff q_0 is (M, P)-generic and*

$$q_0 \Vdash \dot{q}_1 \text{ is } (M[\dot{G}], \dot{Q})\text{-generic.}$$

Proof. Let $(q_0, \dot{q}_1)$ be $(M, P * \dot{Q})$-generic. We first show that q_0 is (M, P)-generic. Let G be V-generic on P such that $q_0 \in G$. It suffices to show that $M[G] \cap \mathrm{Ord} = M \cap \mathrm{Ord}$. Let H be $V[G]$-generic on $\dot{Q}^G$ such that $\dot{q}_1^G \in H$. One can check that

$$G * H = \{(p_0, \dot{p}_1) \in P * \dot{Q} \mid p_0 \in G \text{ and } \dot{p}_1^G \in H\}$$

is V-generic on $P * \dot{Q}$ with $(q_0, \dot{q}_1) \in G * H$. Since $(q_0, \dot{q}_1)$ is $(M, P * \dot{Q})$-generic

$$M \cap \mathrm{Ord} \subseteq M[G] \cap \mathrm{Ord} \subseteq M[G * H] \cap \mathrm{Ord} = M \cap \mathrm{Ord}.$$

To show that

$$q_0 \Vdash \dot{q}_1 \text{ is } (M[\dot{G}], \dot{Q})\text{-generic}$$

it suffices to see that

$$V[G] \vDash \dot{q}_1^G \text{ is } \big(M[G], \dot{Q}^G\big)\text{-generic}.$$

Again take any H being $V[G]$-generic on $\dot{Q}^G$ such that $\dot{q}_1^G \in H$. One has to check that

$$M[G][H] \cap \mathrm{Ord} = M[G] \cap \mathrm{Ord}.$$

$$M[G] \cap \mathrm{Ord} \subseteq M[G][H] \cap \mathrm{Ord} = M[G * H] \cap \mathrm{Ord} = M \cap \mathrm{Ord} \subseteq M[G] \cap \mathrm{Ord}.$$

For the converse assume that q_0 is (M, P)-generic and

$$q_0 \Vdash \dot{q}_1 \text{ is } (M[\dot{G}], \dot{Q})\text{-generic}.$$

Let $G * H$ be V-generic on $P * \dot{Q}$ such that $(q_0, \dot{q}_1) \in G * H$. Then G is V-generic on P such that $q_0 \in G$ and H is $V[G]$-generic on $\dot{Q}^G$ such that $\dot{q}_1^G \in H$. By the assumptions,

$$M[G] \cap \mathrm{Ord} = M \cap \mathrm{Ord} \text{ and } M[G][H] \cap \mathrm{Ord} = M[G] \cap \mathrm{Ord}.$$

Together

$$M[G * H] \cap \mathrm{Ord} = M[G][H] \cap \mathrm{Ord} = M[G] \cap \mathrm{Ord} = M \cap \mathrm{Ord},$$

i.e., $(q_0, \dot{q}_1)$ is $(M, P * \dot{Q})$-generic. □

Lemma 67. *If P is proper and $\Vdash_P \dot{Q}$ is proper then $P * \dot{Q}$ is proper.*

Proof. Let $\lambda > 2^{\mathrm{card}(P)}$ and let $M \prec H_\lambda$ be countable with $P * \dot{Q} \in M$. Let $(p_0, \dot{p}_1) \in P * \dot{Q} \cap M$. By the properness of P take $p \leqslant p_0$ which is (M, P)-generic.

$$p \Vdash \exists q \leqslant \dot{p}_1\, q \text{ is } \big(M\big[\dot{G}_P\big], \dot{Q}\big)\text{-generic}.$$

Take $q_0 \leqslant p$ and $\dot{q}_1 \in \mathrm{dom}(\dot{Q})$ such that

$$q_0 \Vdash \dot{q}_1 \leqslant \dot{p}_1 \text{ amd } \dot{q}_1 \text{ is } \big(M\big[\dot{G}_P\big], \dot{Q}\big)\text{-generic}.$$

By the previous Lemma, $(q_0, \dot{q}_1) \leqslant (p_0, \dot{p}_1)$ is $(M, P * \dot{Q})$-generic. $\square$

13 Countable support iterations

- Problems with finite support iterations: adding Cohen reals through finite supports
- finite supports so far only played a role in chain-condition arguments

We now define *countable support iterations*:

Theorem 68. *Let M be a ground model, and let $((\dot{Q}_\beta, \dot{\leqslant}_\beta)|\beta < \kappa) \in M$ with the property that $\forall \beta < \kappa: \emptyset \in \mathrm{dom}(\dot{Q}_\beta)$. Then there is a uniquely determined sequence $((P_\alpha, \leqslant_\alpha, 1_\alpha)|\alpha \leqslant \kappa) \in M$ such that*

a) $(P_\alpha, \leqslant_\alpha, 1_\alpha)$ is a partial order which consists of α-sequences;

b) $P_0 = \{\emptyset\}$, $\leqslant_0 = \{(\emptyset, \emptyset)\}$, $1_0 = \emptyset$;

c) If $\lambda \leqslant \kappa$ is a limit ordinal then the forcing P_λ is defined by:

$$\begin{aligned} P_\lambda &= \{p: \lambda \to V | (\forall \gamma < \lambda: p \restriction \gamma \in P_\gamma) \wedge \mathrm{card}\,\mathrm{supp}(p) < \aleph_1)\} \\ p \leqslant_\lambda q & \text{ iff } \forall \gamma < \lambda: p \restriction \gamma \leqslant_\gamma q \restriction \gamma \\ 1_\lambda &= (\emptyset | \gamma < \lambda) \end{aligned}$$

d) If $\alpha < \kappa$ and $1_\alpha \Vdash_{P_\alpha}$ "$(\dot{Q}_\alpha, \dot{\leqslant}_\alpha, \emptyset)$ is a forcing", then the forcing $P_{\alpha+1}$ is defined by:

$$\begin{aligned} P_{\alpha+1} &= \{p: \alpha+1 \to V \mid p \restriction \alpha \in P_\alpha \wedge p(\alpha) \in \mathrm{dom}(\dot{Q}_\alpha) \wedge p \restriction \alpha \Vdash_{P_\alpha} p(\alpha) \in \dot{Q}_\alpha\} \\ p \leqslant_{\alpha+1} q & \text{ iff } p \restriction \alpha \leqslant_\alpha q \restriction \alpha \wedge p \restriction \alpha \Vdash_{P_\alpha} p(\alpha) \dot{\leqslant}_\alpha q(\alpha) \\ 1_{\alpha+1} &= (\emptyset | \gamma < \alpha+1) \end{aligned}$$

e) If $\alpha < \kappa$ and not *$1_\alpha \Vdash_{P_\alpha}$ "$(\dot{Q}_\alpha, \dot{\leqslant}_\alpha, \emptyset)$ is a forcing", then the forcing $P_{\alpha+1}$ is defined by:*

$$\begin{aligned} P_{\alpha+1} &= \{p: \alpha+1 \to V \mid p \restriction \alpha \in P_\alpha \wedge p(\alpha) = \emptyset\} \\ p \leqslant_{\alpha+1} q & \text{ iff } p \restriction \alpha \leqslant_\alpha q \restriction \alpha \\ 1_{\alpha+1} &= (\emptyset | \gamma < \alpha+1) \end{aligned}$$

$((P_\alpha, \leqslant_\alpha, 1_\alpha)|\alpha \leqslant \kappa)$, and in particular P_κ are called the *countable support iteration* or *CS-iteration* of the sequence $((\dot{Q}_\beta, \dot{\leqslant}_\beta)|\beta < \kappa)$.

Proof. Exactly as in the finite support case. $\square$

Let us fix a ground model M and the *countable support* iteration $((\dot{Q}_\beta, \dot{\leqslant}_\beta)|\beta < \kappa) \in M$ and $((P_\alpha, \leqslant_\alpha, 1_\alpha)|\alpha \leqslant \kappa) \in M$ as above. Let G_κ be M-generic for P_κ. We analyse the generic extension $M_\kappa = M[G_\kappa]$ by an ascending chain

$$M = M_0 \subseteq M_1 = M[G_1] = M_0[H_0] \subseteq M_2 = M[G_2] = M_1[H_1] \subseteq \ldots \subseteq M_\alpha = M[G_\alpha] \subseteq \ldots \subseteq M_\kappa$$

of generic extensions just like before.

Lemma 69.

a) Let $\alpha \leqslant \kappa$ and $p, q \in P_\alpha$.
Then $p \leqslant_\alpha q$ iff $\forall \gamma \in \mathrm{supp}(p) \cup \mathrm{supp}(q)$: $p \upharpoonright \gamma \Vdash_{P_\gamma} p(\gamma) \dot{\leqslant}_\gamma q(\gamma)$.

b) Let $\alpha \leqslant \beta \leqslant \kappa$ and $p \in P_\beta$. Then $p \upharpoonright \alpha \in P_\alpha$.

c) Let $\alpha \leqslant \beta \leqslant \kappa$ and $p \leqslant_\beta q$. Then $p \upharpoonright \alpha \leqslant_\alpha q \upharpoonright \alpha$.

d) Let $\alpha \leqslant \beta \leqslant \kappa$, $q \in P_\beta$, $\bar{p} \leqslant_\alpha q \upharpoonright \alpha$. Then $\bar{p} \cup (q(\gamma)|\alpha \leqslant \gamma < \beta) \in P_\beta$ and $\bar{p} \cup (q(\gamma)|\alpha \leqslant \gamma < \beta) \leqslant_\beta q$.

For $\alpha \leqslant \kappa$ define $G_\alpha = \{p \upharpoonright \alpha \,|\, p \in G_\kappa\}$.
(1) G_α is M-generic for P_α.

So $M_\alpha = M[G_\alpha]$ is a welldefined generic extension of M by G_α.
(2) Let $\alpha < \beta \leqslant \kappa$. Then $G_\alpha \in M[G_\beta]$ and $M[G_\alpha] \subseteq M[G_\beta]$.

For $\alpha < \kappa$ define

$$Q_\alpha = (Q_\alpha, \leqslant^{Q_\alpha}, \emptyset) = \begin{cases} (\dot{Q}_\alpha^{G_\alpha}, \dot{\leqslant}_\alpha^{G_\alpha}, \emptyset), \text{ if } 1_\alpha \Vdash_{P_\alpha} \text{“}(\dot{Q}_\alpha, \dot{\leqslant}_\alpha, \emptyset) \text{ is a forcing”} \\ (\{\emptyset\}, \{(\emptyset, \emptyset)\}, \emptyset), \text{ else} \end{cases}$$

Then $Q_\alpha \in M_\alpha = M[G_\alpha]$ is a forcing. For $\alpha < \kappa$ define

$$H_\alpha = \{p(\alpha)^{G_\alpha} | p \in G_\kappa\}.$$

(3) H_α is M_α-generic for Q_α.
(4) $M_{\alpha+1} = M_\alpha[H_\alpha]$.
As in the finite support case there are canonical maps between the P_α's. For $\alpha \leqslant \beta \leqslant \kappa$ define $\pi_{\alpha\beta}: P_\alpha \to P_\beta$ by

$$\pi_{\alpha\beta}(p) = p \cup (\emptyset \,|\, \alpha \leqslant \gamma < \beta).$$

Also define $\pi_{\beta\alpha}: P_\beta \rightarrow P_\alpha$ by $\pi_{\beta\alpha}(q) = q \upharpoonright \alpha$. $\pi_{\beta\alpha}$ is a left inverse of $\pi_{\alpha\beta}$:

$$\pi_{\beta\alpha} \circ \pi_{\alpha\beta} = \mathrm{id}_{P_\alpha}.$$

Let the previous constructions take place within a ground model M. Let G_κ be M-generic for P_κ and let $M_\alpha = M[G_\alpha]$ for $\alpha \leqslant \kappa$ be the associated tower of extensions. Let $\alpha \leqslant \beta \leqslant \kappa$. The inclusion $M[G_\alpha] \subseteq M[G_\beta]$ corresponds to the following

Lemma 70. *Let $\dot{x} \in M^{P_\alpha}$ be a P_α-name and $\ddot{x} = \pi^*_{\alpha\beta}(\dot{x}) \in M^{P_\beta}$ its "lift" to P_β. Then*

$$\dot{x}^{G_\alpha} = \ddot{x}^{G_\beta}.$$

14 CS iterations preserve properness

Theorem 71. *Let $(P_\alpha | \, \alpha \leqslant \gamma)$ be a countable support iteration of proper forcings $\big(\dot{Q}_\alpha | \, \alpha < \gamma\big)$, i.e.,*

$$1_\alpha \Vdash \dot{Q}_\alpha \text{ is proper.}$$

Let λ be a sufficiently large cardinal, let $M \prec H_\lambda$ be a countable elementary substructure with $\big(\dot{Q}_\alpha | \, \alpha < \gamma\big) \in M$.

Assume that $\gamma_0 \in \gamma \cap M$, q_0 is (M, P_{γ_0})-generic, $\dot{p}_0 \in V^{P_{\gamma_0}}$ is a P_{γ_0}-name such that

$$(1) \qquad q_0 \Vdash_{P_{\gamma_0}} \dot{p}_0 \in \check{M} \cap \check{P}_\gamma \wedge \dot{p}_0 \upharpoonright \check{\gamma}_0 \in \dot{G}_{\gamma_0}.$$

Then there is a q is (M, P_γ)-generic, such that $q \supseteq q_0$ and

$$(2) \qquad q \Vdash_{P_\gamma} \pi^*_{\gamma_0\,\gamma}(\dot{p}_0) \in \check{M} \cap \check{P}_\gamma \wedge \pi^*_{\gamma_0\,\gamma}(\dot{p}_0) \in \dot{G}_\gamma.$$

Moreover, P_γ is proper.

Notational conventions: this theorem involves canonical shiftings of names between name spaces by maps like $\pi^*_{\gamma_0\,\gamma}$. Also canonical names $\check{x}$ for ground model elements are used frequently. To simplify notation, we usually leave out the π^*'s and $\check{}$'s. So the main implication of the theorem becomes:

Assume that $\gamma_0 \in \gamma \cap M$, q_0 is (M, P_{γ_0})-generic, $\dot{p}_0 \in V^{P_{\gamma_0}}$ such that

$$(1) \qquad q_0 \Vdash_{P_{\gamma_0}} \dot{p}_0 \in M \cap P_\gamma \wedge \dot{p}_0 \upharpoonright \gamma_0 \in \dot{G}_{\gamma_0}.$$

Then there is a q is (M, P_γ)-generic, such that $q \supseteq q_0$ and

$$(2) \qquad\qquad q \Vdash_{P_\gamma} \dot{p}_0 \in M \cap P_\gamma \wedge \dot{p}_0 \in \dot{G}_\gamma\,.$$

Proof. By induction on the length γ of the iteration.

For $\gamma = 0$ the theorem is trivial.

Let $\gamma = \gamma_1 + 1$ be a successor ordinal and λ, M as above. Let γ_0, q_0, $\dot{p}_0$ be as above so that (1) holds.

First consider the case $\gamma_0 < \gamma_1$. Restricting (1) to γ_1 yields

$$(1') \qquad\qquad q_0 \Vdash_{P_{\gamma_0}} \dot{p}_0 \upharpoonright \gamma_1 \in M \cap P_{\gamma_1} \wedge (\dot{p}_0 \upharpoonright \gamma_1) \upharpoonright \gamma_0 \in \dot{G}_{\gamma_0}\,.$$

Applying the Theorem inductively at $\gamma_1 < \gamma$ yields q_1 (M, P_{γ_1})-generic, such that $q_1 \supseteq q_0$ *and*

$$(2') \qquad\qquad q_1 \Vdash_{P_\gamma} \dot{p}_0 \upharpoonright \gamma_1 \in M \cap P_{\gamma_1} \wedge \dot{p}_0 \upharpoonright \gamma_1 \in \dot{G}_{\gamma_1}\,.$$

This is property (1) in the special case $\gamma_0 = \gamma_1$.

So it suffices to obtain (2) from (1) in the special case that $\gamma = \gamma_0 + 1$. By the setup of the iteration

$$q_0 \Vdash_{P_{\gamma_0}} \dot{Q}_{\gamma_0} \text{ is proper and } \dot{p}_0(\gamma_0) \in M\big[\dot{G}_{\gamma_0}\big] \cap \dot{Q}_{\gamma_0}\,.$$

Then q_0 forces the existence of an $\big(M\big[\dot{G}_{\gamma_0}\big],\ \dot{Q}_{\gamma_0}\big)$-generic condition $\leqslant_{\gamma_0} \dot{p}_0(\gamma_0)$. By the maximality principle there is a P_{γ_0}-name $\dot{q}$ such that

$$q_0 \Vdash_{P_{\gamma_0}} \dot{q} \text{ is } \big(M\big[\dot{G}_{\gamma_0}\big], \dot{Q}_{\gamma_0}\big)\text{-generic and } \dot{q} \mathrel{\dot{\leqslant}}_{\gamma_0} \dot{p}_0(\gamma_0).$$

Set $q = q_0{}^\frown \dot{q} \in P_{\gamma_0+1} = P_\gamma$. By a previous lemma, q is (M, P_γ)-generic. Since $q \leqslant_\gamma q_0{}^\frown\emptyset$ we have

$$q \Vdash_{P_\gamma} \dot{p}_0 \in M \cap P_\gamma\,.$$

It remains to show that $q \Vdash_{P_\gamma} \dot{p}_0 \in \dot{G}_\gamma$. Let $q' \leqslant_\gamma q$ such that $q' \Vdash \dot{p}_0 = \check{p}$ for some $p \in M \cap P_\gamma$. Since $\dot{p}_0$ is (originally) a P_{γ_0}-name already $q' \upharpoonright \gamma_0 \Vdash \dot{p}_0 = \check{p}$ and, by (1),

$$q' \upharpoonright \gamma_0 \Vdash \check{p} \upharpoonright \gamma_0 = \dot{p}_0 \upharpoonright \gamma_0 \in \dot{G}_{\gamma_0}\,.$$

Since P_{γ_0} is a separative forcing this implies

$$q' \upharpoonright \gamma_0 \leqslant_{\gamma_0} p \upharpoonright \gamma_0\,.$$

Also

$$q' \upharpoonright \gamma_0 \Vdash_{P_{\gamma_0}} q'(\gamma_0) \dot{\leqslant}_{\gamma_0} q(\gamma_0) = \dot{q} \dot{\leqslant}_{\gamma_0} \dot{p}_0(\gamma_0) = \check{p}(\gamma_0).$$

Thus $q' \leqslant_\gamma p$ and $q' \Vdash \check{p} \in \dot{G}_\gamma$. Hence

$$q' \Vdash \dot{p}_0 \in \dot{G}_\gamma .$$

Since there are densely many such $q' \leqslant_\gamma q$ we get

$$q \Vdash_{P_\gamma} \dot{p}_0 \in \dot{G}_\gamma,$$

and (2) is satisfied.

Now let γ be a limit ordinal. Assume that $\gamma_0 \in \gamma \cap M$, q_0 is (M, P_{γ_0})-generic, $\dot{p}_0 \in V^{P_{\gamma_0}}$ is a P_{γ_0}-name such that

$$(1) \qquad q_0 \Vdash_{P_{\gamma_0}} \dot{p}_0 \in M \cap P_\gamma \wedge \dot{p}_0 \upharpoonright \gamma_0 \in \dot{G}_{\gamma_0} .$$

Let $(\gamma_n | \, n \in \omega)$ be a strictly increasing sequence of ordinals $\gamma_n \in \gamma \cap M$, starting with the given γ_0, which is cofinal in the countable set $\gamma \cap M$. To obtain genericity in the construction, let $(D_n | \, n \in \omega)$ be an enumeration of the $D \in M$ which are dense in P_γ. Starting with the given q_0 and $\dot{p}_0$ construct sequences $(q_n | \, n \in \omega)$ and $(\dot{p}_n | \, n \in \omega)$ such that for $n \geqslant 1$

(1) q_n is an (M, P_{γ_n})-generic condition and $q_n \upharpoonright \gamma_{n-1} = q_{n-1}$

(2) $\dot{p}_n$ is a P_{γ_n}-name such that for $n \geqslant 1$

$$q_n \Vdash_{P_{\gamma_n}} \dot{p}_n \in M \cap P_\gamma \wedge \dot{p}_n \upharpoonright \gamma_n \in \dot{G}_{\gamma_n} \wedge \dot{p}_n \leqslant_\gamma \dot{p}_{n-1} \wedge \dot{p}_n \in D_{n-1} .$$

Suppose that $q_n, \dot{p}_n$ with these properties are defined. To proceed to $q_{n+1}, \dot{p}_{n+1}$ with these properties, consider a V-generic filter $G_{\gamma_n} \subseteq P_{\gamma_n}$ with $q_n \in G_{\gamma_n}$. Let

$$p_n = \dot{p}_n^{G_{\gamma_n}} \in M \cap P_\gamma .$$

Note that $p_n \upharpoonright \gamma_n \in G_{\gamma_n}$ since $q_n \Vdash_{P_{\gamma_n}} \dot{p}_n \upharpoonright \gamma_n \in \dot{G}_{\gamma_n}$.

The set

$$D = \{ d \upharpoonright \gamma_n | \, d \leqslant_\gamma p_n \wedge d \in D_n \} \cup \{ d \in P_{\gamma_n} | \, d \perp p_n \upharpoonright \gamma_n \}$$

is defined from parameters in M, hence $D \in M$.

Claim. D is dense in P_{γ_n}.

Proof. Let $r \in P_{\gamma_n}$. We may assume that r is compatible with $p_n \restriction \gamma_n$. Let $s \leqslant_{\gamma_n} r, p_n \restriction \gamma_n$. Then $s^\frown(p_n(i)| \gamma_n \leqslant i < \gamma) \leqslant_\gamma p_n$. By the density of D_n take $d \in D_n$ such that

$$d \leqslant_{\gamma_n} s^\frown(p_n(i)| \gamma_n \leqslant i < \gamma).$$

Then $d \restriction \gamma_n \in D$ and $d \restriction \gamma_n \leqslant_{\gamma_n} s \leqslant_{\gamma_n} r$. *qed(Claim)*

Since q_n is (M, P_{γ_n})-generic, $M \cap D$ is predense below q_n and the generic filter G_{γ_n} meets $M \cap D$: take

$$r \in M \cap D \cap G_{\gamma_n}.$$

Then $r, p_n \restriction \gamma_n \in G_{\gamma_n}$ are compatible and so $r = d \restriction \gamma_n$ for some $d \leqslant_\gamma p_n$ with $d \in D_n$. Since $r, p_n, D_n \in M$ one can find such d in M:

$$V[G_{\gamma_n}] \vDash \exists d\, (d \in M \cap P_\gamma \wedge d \restriction \gamma_n \in G_{\gamma_n} \wedge d \leqslant_\gamma p_n \wedge d \in D_n).$$

By the maximality principle there is a P_{γ_n}-name $\dot{p}_{n+1}$ such that

$$q_n \Vdash \dot{p}_{n+1} \in M \cap P_\gamma \wedge \dot{p}_{n+1} \restriction \gamma_n \in \dot{G}_{\gamma_n} \wedge \dot{p}_{n+1} \leqslant_\gamma \dot{p}_n \wedge \dot{p}_{n+1} \in D_n.$$

Now we apply the inductive hypothesis at $\gamma_{n+1} < \gamma$ to the property

$$q_n \Vdash \dot{p}_{n+1} \restriction \gamma_{n+1} \in M \cap P_{\gamma_{n+1}} \wedge (\dot{p}_{n+1} \restriction \gamma_{n+1}) \restriction \gamma_n \in \dot{G}_{\gamma_n}.$$

We find an $(M, P_{\gamma_{n+1}})$-generic condition $q_{n+1} \supseteq q_n$ such that

$$q_{n+1} \Vdash_{P_{\gamma_{n+1}}} \dot{p}_{n+1} \restriction \gamma_{n+1} \in \dot{G}_{\gamma_{n+1}}.$$

Together we have

$$q_{n+1} \Vdash \dot{p}_{n+1} \in M \cap P_\gamma \wedge \dot{p}_{n+1} \restriction \gamma_{n+1} \in \dot{G}_{\gamma_{n+1}} \wedge \dot{p}_{n+1} \leqslant_\gamma \dot{p}_n \wedge \dot{p}_{n+1} \in D_n.$$

This completes the recursive construction of the sequences $(q_n| n \in \omega)$ and $(\dot{p}_n| n \in \omega)$.

Define the desired $q \in P_\gamma$ as

$$q = \left(\bigcup_{n<\omega} q_n\right) \cup (\emptyset| \sup(\gamma \cap M) \leqslant i < \gamma).$$

Obviously, $q \Vdash_{P_\gamma} \dot{p}_0 \in M \cap P_\gamma$. For the proof of (M, P_γ)-genericity we first show

Claim. Let $n \in \omega$. Then $q \Vdash_{P_\gamma} \dot{p}_n \in \dot{G}_\gamma$.

Proof. Let $q' \leqslant_\gamma q$ such that $q' \Vdash \dot{p}_n = \check{p}$ for some $p \in M \cap P_\gamma$. For $m \geqslant n$ we have

$$q' \upharpoonright \gamma_m \Vdash \check{p} \upharpoonright \gamma_m \in \dot{G}_{\gamma_m}$$

since

$$q_m \geqslant_{\gamma_m} q' \upharpoonright \gamma_m \Vdash \dot{p}_m \upharpoonright \gamma_m \in \dot{G}_{\gamma_m} \wedge \dot{p}_m \upharpoonright \gamma_m \leqslant_{\gamma_m} \dot{p}_n \upharpoonright \gamma_m = \check{p}_n \upharpoonright \gamma_m .$$

Since P_{γ_m} is a separative partial order, $q' \upharpoonright \gamma_m \leqslant_{\gamma_m} p \upharpoonright \gamma_m$.

Since $p \in M$, $\operatorname{supp}(p) \subseteq \bigcup_{m<\omega} \gamma_m$. Therefore $q' \leqslant_\gamma p$ which trivially implies

$$q' \Vdash_{P_\gamma} \check{p} \in \dot{G}_\gamma \text{ and } q' \Vdash_{P_\gamma} \dot{p}_n \in \dot{G}_\gamma .$$

Since the set of q' which decide the value of $\dot{p}_n$ is dense below q the Claim holds. *qed(Claim)*

Claim. q is (M, P_γ)-generic.

Proof. Let $n < \omega$. It suffices to show that q is compatible with some element of $D_n \cap M$. Let G_γ be V-generic for P_γ with $q \in G_\gamma$. Let $p = \dot{p}_{n+1}^{G_\gamma}$. By the recursive construction, $q_{n+1} \Vdash \dot{p}_{n+1} \in D_n \cap M$, and thus $p \in D_n \cap M$. By the previous Claim, $p \in G_\gamma$. Thus q is compatible with $p \in D_n \cap M$. *qed(Claim)*

So we can move from (1) to (2) at length γ. We conclude the induction by

Claim. P_γ is proper.

Proof. Let $M \prec H_\lambda$ as above. Let $p_0 \in M \cap P_\gamma$. Property (1) is satisfied with $\gamma_0 = 0$. Trivially

$$(1'') \qquad 1_0 \Vdash_{P_0} \check{p}_0 \in M \cap P_\gamma \wedge \check{p}_0 \upharpoonright 0 \in \dot{G}_0 .$$

The above implication yields an (M, P_γ)-generic $q \supseteq 1_0$ such that

$$(2'') \qquad q \Vdash_{P_\gamma} \check{p}_0 \in M \cap P_\gamma \wedge \check{p}_0 \in \dot{G}_\gamma .$$

Since P_γ is a separative forcing, the generic condition $q \leqslant p_0$, as required. $\square$

15 Supercompact cardinals

In analogy with MA we define

Definition 72. *The* Proper Forcing Axiom (PFA) *postulates* $\mathrm{FA}_{\aleph_1}(P)$ *for every proper forcing P.*

Obviously PFA implies $\mathrm{MA}_{\aleph_1}$ and hence $2^{\aleph_0} > \aleph_1$. A difficult theorem actually shows that PFA implies $2^{\aleph_0} = \aleph_2$. In this lecture we shall prove $\mathrm{Con}(\mathrm{PFA} + 2^{\aleph_0} = \aleph_2)$ from large cardinal hypotheses. It can also be shown that (some) large cardinals are necessary.

The large cardinals will be used for a strong *reflection argument* which in the forcing for $\mathrm{Con}(\mathrm{MA} + 2^{\aleph_0} = \kappa)$ only required that κ was regular and $\geqslant \aleph_2$. For MA we had a Lemma stating: for infinite cardinals ξ the following are equivalent:

a) MA_ξ;

b) for every ccc forcing Q whose underlying set is a subset of ξ and every $\mathcal{D} \subseteq \mathcal{P}(\xi)$ with $\mathrm{card}(\mathcal{D}) \leqslant \xi$ there exists a $\mathcal{D}$-*generic* filter on Q.

So it suffices to deal with all ccc forcings Q below ξ^+. ξ^+ and κ *reflect* the needed requirements below κ. The situation for PFA is more difficult and requires large cardinal type reflections obtained via elementary embeddings.

15.1 $\mathcal{P}_\kappa(\lambda)$

We shall use elementary embeddings π between inner models of set theory, where $\pi \upharpoonright \kappa = \mathrm{id}$ and $\pi(\kappa) > \lambda$. To capture such embeddings combinatorially, i.e., by sets, we work on the following kind of domains:

Definition 73. $\mathcal{P}_\kappa(\lambda) = [\lambda]^{<\kappa} = \{x \subseteq \lambda \mid \mathrm{card}(x) < \kappa\}$.

Definition 74. *U is an* ultrafilter *on $\mathcal{P}_\kappa(\lambda)$ if*

a) $U \subseteq \mathcal{P}(\mathcal{P}_\kappa(\lambda))$

b) $\mathcal{P}_\kappa(\lambda) \in U$, $\emptyset \notin U$

c) $U \ni A \subseteq B \subseteq \mathcal{P}_\kappa(\lambda) \to B \in U$

d) $A, B \in U \to A \cap B \in U$

Definition 75. *An ultrafilter U on $\mathcal{P}_\kappa(\lambda)$ is*

a) κ-complete *if* $\{A_i \mid i < \delta\} \subseteq U$ *with* $\delta < \kappa$ *imply* $\bigcap_{i<\delta} A_i \in U$*;*

b) fine *if for every $x \in \mathcal{P}_\kappa(\lambda)$ the* upper cone $\hat{x} = \{y \in \mathcal{P}_\kappa(\lambda) \mid x \subseteq y\}$ *is an element of U;*

c) normal *if for $h: \mathcal{P}_\kappa(\lambda) \to \lambda$ which is almost everywhere regressive, i.e., $\{x \in \mathcal{P}_\kappa(\lambda) \mid h(x) \in x\} \in U$, there is some $\delta < \lambda$ such that*

$$\{x \in \mathcal{P}_\kappa(\lambda) \mid h(x) = \delta\} \in U.$$

Definition 76. *Let κ be an uncountable cardinal and $\lambda \geqslant \kappa$. Then κ is* λ-supercompact *if there is a κ-complete, fine, normal ultrafilter on $\mathcal{P}_\kappa(\lambda)$. Call such an ultrafilter a λ-supercompact ultrafilter on κ. κ is* supercompact *if κ is supercompact for every $\lambda \geqslant \kappa$.*

One can show straightforwardly that a measurable cardinal κ is κ-supercompact.

15.2 Ultrapowers

Fix a λ-supercompact ultrafilter U on κ. We shall define the *ultrapower* of the universe V by U in the following construction.

Let ${}^*V = {}^{\mathcal{P}_\kappa(\lambda)}V = \{f \mid f : \mathcal{P}_\kappa(\lambda) \to V\}$.

Define an equivalence relation $=_U$ on ${}^*V$ by

$$f =_U g \text{ iff } \{x \mid f(x) = g(x)\} \in U.$$

We represent equivalence classes by sets of elements of lowest von Neumann-degree:

for $f \in {}^*V$ set

$$[f] = \{g \in V_\alpha \mid f =_U g\}$$

where $\alpha \in \mathrm{Ord}$ is minimal such that $V_\alpha \cap \{g \mid f =_U g\} \neq \emptyset$. This technique is called *Scott's trick*.

The *ultrapower* of V by U is

$$\mathrm{Ult}(V, U) = \{[f] \mid f \in {}^*V\}.$$

Define an „$\in$-structure“ on $\mathrm{Ult}(V, U)$ by

$$[f] \in_U [g] \text{ iff } \{x \mid f(x) \in g(x)\} \in U.$$

The „structure“ $(\mathrm{Ult}(V, U), \in_U)$ satisfies *Los's Theorem*

Theorem 77. *For any formula $\varphi(v_0, \ldots, v_{n-1})$ of set theory and all $[f_0], \ldots, [f_{n-1}] \in \mathrm{Ult}(V, U)$*

$$(\mathrm{Ult}(V, U), \in_U) \vDash \varphi([f_0], \ldots, [f_{n-1}]) \textit{ iff } \{x \mid \varphi(f_0(x), \ldots, f_{n-1}(x))\} \in U.$$

Proof. By induction on the complexity of φ. □

Lemma 78. $\mathrm{Ult}(V,U),\in_U$ *is* set-like*, i.e., for every* $f\in^* V$*,* $\{[g]|\,[g]\in_U[f]\}\in V$*.*

Proof. Let $f\in V_\alpha$. Let $[g]\in_U[f]$. Then

$$A=\{x|\ g(x)\in f(x)\}\in U.$$

Define $g'\in^* V$ by

$$g'(x)=\begin{cases} g(x), \text{ if } x\in A\\ 0, \text{ else}\end{cases}$$

Then $[g]=[g']$ and $g'\colon\mathcal{P}_\kappa(\lambda)\to V_\alpha$. Then $\{[g]|\,[g]\in_U[f]\}$ is a set since

$$\{[g]|\,[g]\in_U[f]\}\subseteq\{[g']|\,g'\colon\mathcal{P}_\kappa(\lambda)\to V_\alpha\}\in V.$$

□

Lemma 79. $\mathrm{Ult}(V,U),\in_U$ *is wellfounded.*

Proof. Assume not. Then there are $f_0,f_1,\ldots\in^* V$ such that for $n<\omega$

$$[f_{n+1}]\in_U[f_n].$$

Let $A_n=\{x|\ f_{n+1}(x)\in f_n(x)\}\in U$. By the κ-completeness of U take $x_0\in\bigcap_{n<\omega}A_n$. Then

$$f_{n+1}(x_0)\in f_n(x_0)$$

for $n<\omega$ which contradicts foundation. □

Theorem 80. $\mathrm{Ult}(V,U),\in_U$ *is a model of the* ZFC *axioms.*

Proof. Let φ be an axiom of ZFC. Hence φ holds (in V) and

$$\{x\in\mathcal{P}_\kappa(\lambda)|\,\varphi\}=\mathcal{P}_\kappa(\lambda)\in U.$$

By Los' Theorem,

$$\mathrm{Ult}(V,U),\in_U\vDash\varphi.$$

□

In particular, $\mathrm{Ult}(V,U)$, $\in_U$ satisfies the axiom of extensionality and there is a uniquely defined isomorphism

$$\sigma\colon(\mathrm{Ult}(V,U),\in_U)\cong(M,\in)$$

where M is transitive.

$(M, \in)$ is a class-sized transitive model of ZFC, hence it is an inner model. Many notions are absolute between M and V.

Every element of M is of the form $\sigma([f])$ for some function $f: \mathcal{P}_\kappa(\lambda) \to V$; f is a representative of $\sigma([f])$ in the ultrapower. We also write $\sigma[f]$ instead of $\sigma([f])$. We consider specific representative functions.

For $\gamma \in \mathrm{Ord}$ define the function $k_\gamma: \mathcal{P}_\kappa(\lambda) \to \mathrm{Ord}$ by

$$k_\gamma(x) = \mathrm{otp}(\gamma \cap x).$$

Lemma 81. *Let $\gamma \leqslant \lambda$. Then $\sigma[k_\gamma] = \gamma$.*

Proof. By induction on γ.

(1) $\sigma[k_\gamma]$ is an ordinal, since $\{x \in \mathcal{P}_\kappa(\lambda) | k_\gamma(x) \in \mathrm{Ord}\} = \mathcal{P}_\kappa(\lambda) \in U$.

(2) $\sigma[k_\gamma] \geqslant \gamma$.

Proof. Let $\delta < \gamma$. By Los' theorem $[k_\delta] \in_U [k_\gamma]$ since

$$\{x | k_\delta \in k_\gamma\} = \{x | \mathrm{otp}(\delta \cap x) < \mathrm{otp}(\gamma \cap x)\} \supseteq \{x | \delta \in x\} = \widehat{\{x\}} \in U.$$

By induction hypothesis, $\delta = \sigma[k_\delta] \in \sigma[k_\gamma]$.

qed(1)

(3) $\sigma[k_\gamma] = \gamma$.

Proof. Assume instead $\sigma[k_\gamma] > \gamma$. Take $f \in^* V$ such that $\sigma[f] = \gamma$. Then $[f] \in_U [k_\gamma]$ and

$$A = \{x | f(x) \in \mathrm{otp}(\gamma \cap x)\} \in U.$$

Define $h: A \to \lambda$ by $h(x) =$ the $f(x)^{\mathrm{th}}$ element of x. h is regressive. By the normality of U there is some $\delta < \lambda$ such that

$$B = \{x \in A | h(x) = \delta\} \in U.$$

For $x \in B$ observe that $\delta =$ the $f(x)^{\mathrm{th}}$ element of x, i.e.,

$$f(x) = \mathrm{otp}(\delta \cap x) = k_\delta(x).$$

Hence, by induction,

$$\sigma[f] = \sigma[k_\delta] = \delta,$$

contradicting $\sigma[f] = \gamma$. □

There are other canonical functions in ${}^*V$. For $a \in V$ define the function $\mathrm{const}_a \colon \mathcal{P}_\kappa(\lambda) \to V$ by

$$\mathrm{const}_a(x) = a.$$

Lemma 82.

a) Let $\gamma < \kappa$. Then $\sigma[\mathrm{const}_\gamma] = \gamma$.

b) $\sigma[\mathrm{const}_\kappa] > \lambda$.

Proof. *a)* If $\gamma \subseteq x \in \mathcal{P}_\kappa(\lambda)$ then $k_\gamma(x) = \mathrm{otp}(\gamma \cap x) = \gamma = \mathrm{const}_\gamma(x)$. Since U is fine, $\sigma[\mathrm{const}_\gamma] = \sigma[k_\gamma] = \gamma$.

b) For any $x \in \mathcal{P}_\kappa(\lambda)$, $\mathrm{const}_\kappa(x) = \kappa > \mathrm{otp}(x) = \mathrm{otp}(\lambda \cap x) = k_\lambda(x)$. Hence $\sigma[\mathrm{const}_\kappa] > \sigma[k_\lambda] = \lambda$. □

The model M is closed w.r.t λ-sequences:

Lemma 83. *${}^\lambda M \subseteq M$.*

Proof. Consider $(\sigma[g_\gamma] |\ \gamma < \lambda)$, where $g_\gamma \in^* V$. To represent this sequence in the ultrapower, define $G \colon \mathcal{P}_\kappa(\lambda) \to V$ by

$$G(x) = (g_{x(i)}(x) |\ i \in \mathrm{otp}(x))$$

where $x(i)$ denotes the i-th element of x in its canonical enumeration. For every $x \in \mathcal{P}_\kappa(\lambda)$ we have

$$G(x) \text{ is a sequence of length } \mathrm{otp}(x) = k_\lambda(x).$$

By the theorem of Los, $\sigma[G] \in M$ is a sequence of length $\sigma[k_\lambda] = \lambda$. To show that $\sigma[G]$ is the given λ-sequence, consider $\gamma < \lambda$. For $x \in \mathcal{P}_\kappa(\lambda)$ with $\gamma \in x$ we have

$$\gamma = \text{ the } k_\gamma(x)\text{-th element of } x = x(k_\gamma(x))$$

and

$$G(x)(k_\gamma(x)) = g_{x(k_\gamma(x))}(x) = g_\gamma(x).$$

By Los' theorem,

$$\sigma[G](\gamma) = \sigma[G](\sigma[k_\gamma]) = \sigma[g_\gamma].$$

□

Definition 84. *The ultrapower embedding* $\pi: V \to M$ *is defined by*

$$\pi(a) = \sigma[\mathrm{const}_a].$$

Theorem 85.

a) $\pi: (V, \in) \to (M, \in)$ *is an elementary embedding, i.e., for every* $\in$*-formula* $\varphi(v_0, \ldots, v_{n-1})$ *and every* $a_0, \ldots, a_{n-1} \in V$

$$\varphi(a_0, \ldots, a_{n-1}) \text{ iff } (M, \in) \models \varphi(\pi(a_0), \ldots, \pi(a_{n-1})).$$

b) π *has* critical point κ, *i.e.,* $\pi \upharpoonright \kappa = \mathrm{id} \upharpoonright \kappa$ *and* $\pi(\kappa) > \kappa$. *Indeed* $\pi(\kappa) > \lambda$.

Proof. *a)*

$$\begin{aligned}
\varphi(a_0, \ldots, a_{n-1}) &\leftrightarrow \{x \in \mathcal{P}_\kappa(\lambda) | \, \varphi(\mathrm{const}_{a_0}(x), \ldots, \mathrm{const}_{a_{n-1}}(x))\} = \mathcal{P}_\kappa(\lambda) \\
&\leftrightarrow \{x \in \mathcal{P}_\kappa(\lambda) | \, \varphi(\mathrm{const}_{a_0}(x), \ldots, \mathrm{const}_{a_{n-1}}(x))\} \in U \\
&\leftrightarrow (M, \in) \models \varphi(\sigma[\mathrm{const}_{a_0}], \ldots, \sigma[\mathrm{const}_{a_{n-1}}]) \\
&\leftrightarrow (M, \in) \models \varphi(\pi(a_0), \ldots, \pi(a_{n-1})).
\end{aligned}$$

b) Let $\gamma < \kappa$. Then

$$\pi(\gamma) = \sigma[\mathrm{const}_\gamma] = \sigma[k_\gamma] = \gamma \text{ and } \pi(\kappa) = \sigma[\mathrm{const}_\kappa] > \lambda. \qquad \square$$

The elementary embedding was defined from the ultrafilter. Conversely, the ultrafilter is definable from the embedding:

Lemma 86.

a) *Define* $d: \mathcal{P}_\kappa(\lambda) \to \mathcal{P}_\kappa(\lambda)$ *by* $d(x) = x$. *Then*

$$\sigma[d] = \pi''\lambda$$

b) *For* $A \subseteq \mathcal{P}_\kappa(\lambda)$

$$A \in U \text{ iff } \pi''\lambda \in \pi(A).$$

Proof. *a)* ($\subseteq$) Let $\sigma[f] \in \sigma[d]$. Then $\{x | \, f(x) \in d(x) = x\} \in U$, i.e., f is regressive on an ultrafilter set. By the normality of U there is some $\gamma < \lambda$ such that $\{x | \, f(x) = \gamma\} \in U$. Then $\{x | \, f(x) = \mathrm{const}_\gamma(x)\} \in U$ and by the theorem of Los,

$$\sigma[f] = \sigma[\mathrm{const}_\gamma] = \pi(\gamma) \in \pi''\lambda.$$

($\supseteq$) Let $\gamma < \lambda$. Then

$$\{x |\, \mathrm{const}_\gamma(x) \in d(x)\} = \{x |\, \gamma \in x\} \in U,$$

and so

$$\pi(\gamma) = \sigma[\mathrm{const}_\gamma] \in \sigma[d].$$

b)

$$\begin{aligned} A = \{x |\, x \in A\} \in U &\leftrightarrow \{x |\, d(x) \in \mathrm{const}_A(x)\} \in U \\ &\leftrightarrow \sigma[d] \in \sigma[\mathrm{const}_A] \\ &\leftrightarrow \pi''\lambda \in \pi(A) \end{aligned}$$

□

Lemma 87. *Let U be a κ-complete, fine, normal ultrafilter on $\mathcal{P}_\kappa(\lambda)$ and let $\kappa \leqslant \bar{\lambda} < \lambda$. For $x \in \mathcal{P}_\kappa(\lambda)$ define $\bar{x} = x \cap \bar{\lambda}$. For $A \subseteq \mathcal{P}_\kappa(\lambda)$ define $\bar{A} = \{\bar{x} |\, x \in A\}$, and finally*

$$U \upharpoonright \bar{\lambda} = \bar{U} = \{\bar{A} |\, A \in U\}.$$

Then $\bar{U}$ is a κ-complete, fine, normal ultrafilter on $\mathcal{P}_\kappa(\bar{\lambda})$.

Proof. (1) Let $A \in U$ and $\bar{A} \subseteq B \subseteq \mathcal{P}_\kappa(\bar{\lambda})$. Then $A \cup B \in U$ and

$$B = \bar{A} \cup B = \{\bar{x} |\, x \in A\} \cup \{\bar{x} |\, x \in B\} = \{\bar{x} |\, x \in A \cup B\} \in \bar{U}.$$

(2) Let $A, B \in U$. Then $\bar{A} \cap \bar{B} \in \bar{U}$ since

$$\bar{A} \cap \bar{B} = \{\bar{x} |\, x \in A\} \cap \{\bar{x} |\, x \in B\} \supseteq \{\bar{x} |\, x \in A \cap B\} \in \bar{U}.$$

(3) $\mathcal{P}_\kappa(\bar{\lambda}) = \overline{\mathcal{P}_\kappa(\lambda)} \in \bar{U}$.

(4) $\emptyset \notin \bar{U}$: indeed, if $A \in U$ then $A \neq \emptyset$ and $\bar{A} = \{\bar{x} |\, x \in A\} \neq \emptyset$.

(5) $\bar{U}$ is κ-complete. *Proof*. Let $\{A_i |\, i < \delta\} \subseteq U$ where $\delta < \kappa$. Then $\bigcap_{i<\delta} A_i \in U$, and $\bigcap_{i<\delta} \bar{A}_i \in \bar{U}$ since

$$\bigcap_{i<\delta} \bar{A}_i = \bigcap_{i<\delta} \{\bar{x} |\, x \in A_i\} \supseteq \left\{\bar{x} |\, x \in \bigcap_{i<\delta} A_i\right\} \in \bar{U}.$$

(6) $\bar{U}$ is fine. *Proof*. Let $z \in \mathcal{P}_\kappa(\bar{\lambda})$. Then

$$\hat{z} = \{x \in \mathcal{P}_\kappa(\bar{\lambda}) |\, x \supseteq z\} = \{\bar{x} |\, x \in \mathcal{P}_\kappa(\lambda) \wedge x \supseteq z\} \in \bar{U}$$

since U is fine.

(7) $\bar{U}$ is normal. *Proof*. Let $A \in U$ and let $\bar{h}: \bar{A} \to \bar{\lambda}$ be regressive. We may assume that $\emptyset \notin \bar{A}$. Define a regressive $h: A \to \bar{\lambda}$ by $h(x) = \bar{h}(\bar{x})$. By the normality of U there is $\delta < \bar{\lambda}$ such that $B = \{x \in A | \, h(x) = \delta\} \in U$. Then

$$\{y \in \bar{A} | \, \bar{h}(y) = \delta\} = \{\bar{x} | \, x \in A \wedge \bar{h}(\bar{x}) = h(x) = \delta\} = \bar{B} \in \bar{U}. \qquad \square$$

For a λ-supercompact ultrafilter U on κ let π_U be the ultrapower embedding defined from U. Also denote the transitivisation of $\mathrm{Ult}(V, U)$ by M_U : $\sigma_U: (\mathrm{Ult}(V, U), \in_U) \cong (M_U, \in)$.

Lemma 88. *In the situation of the previous Lemma define a map* $\rho: M_{\bar{U}} \to M_U$ *by*

$$\rho(\sigma_{\bar{U}}[f]) = \sigma_U[f']$$

where $f': \mathcal{P}_\kappa(\lambda) \to V$ *is defined from* $f: \mathcal{P}_\kappa(\bar{\lambda}) \to V$ *by*

$$f'(x) = f(\bar{x}) = f(x \cap \bar{\lambda}).$$

Then $\rho: (M_{\bar{U}}, \in) \to (M_U, \in)$ *is elementary with*

$$\pi_U = \rho \circ \pi_{\bar{U}}.$$

Moreover $\rho \upharpoonright (\bar{\lambda} + 1) = \mathrm{id}$. $\rho \upharpoonright H_{\bar{\lambda}^+} = \mathrm{id}$.

Proof. Let $\varphi(v_0, \ldots, v_{n-1})$ be an $\in$-formula and $\sigma_{\bar{U}}[f_0], \ldots \in M_{\bar{U}}$. Then

$$\begin{array}{rl} (M_{\bar{U}}, \in) \models \varphi(\sigma_{\bar{U}}[f_0], \ldots) & \text{iff} \\ \{x \in \mathcal{P}_\kappa(\bar{\lambda}) | \, \varphi(f_0(x), \ldots)\} \in \bar{U} & \text{iff} \\ \{x \in \mathcal{P}_\kappa(\lambda) | \, \varphi(f_0'(x), \ldots)\} \in U & \text{iff} \\ (M_U, \in) \models \varphi(\sigma_U[f_0'], \ldots) & \end{array}$$

Let $a \in V$. Then

$$\rho \circ \pi_{\bar{U}}(a) = \rho(\sigma_{\bar{U}}[\mathrm{const}_a]) = \sigma_U[\mathrm{const}_a'] = \pi_U(a)$$

since const_a' is the corresponding constant function on $\mathcal{P}_\kappa(\lambda)$.

For $\delta \leqslant \bar{\lambda}$ consider the representing function $k_\delta: \mathcal{P}_\kappa(\bar{\lambda}) \to \kappa$, $k_\delta(x) = \mathrm{otp}(\delta \cap x)$. Then for $x \in \mathcal{P}_\kappa(\lambda)$ we have

$$k_\delta'(x) = k_\delta(x \cap \bar{\lambda}) = \mathrm{otp}(\delta \cap (x \cap \bar{\lambda})) = \mathrm{otp}(\delta \cap x)$$

which is the function representing δ in $\mathrm{Ult}(V, U)$. Hence

$$\rho(\delta) = \rho(\sigma_{\bar{U}}[k_\delta]) = \sigma_U[k_\delta'] = \delta. \qquad \square$$

Theorem 89. *Let* κ *be supercompact. Then there exists a* Laver function $f: \kappa \to V_\kappa$ *such that for every* x *and every* $\lambda \geqslant \kappa$ *with* $\lambda \geqslant \mathrm{card}(\mathrm{TC}(x))$ *there exists a* λ*-supercompact ultrafilter* U *on* κ *such that* $\pi_U(f)(\kappa) = x$.