PROBLEM SET 11 - MODEL THEORY

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Problem 47 (4 points)**.** Let $\langle X, cl\rangle$ be a pregeometry and let $A \subseteq X$. Define

$$cl^A : \mathcal{P}(A) \longrightarrow \mathcal{P}(A);\ Y \longmapsto A \cap cl(Y)$$

and

$$cl_A : \mathcal{P}(X) \longrightarrow \mathcal{P}(X);\ Z \longmapsto cl(A \cup Z).$$

Prove the following statements:

(1) $\langle A, cl^A\rangle$ is a pregeometry.
(2) $\langle X, cl_A\rangle$ is a pregeometry.
(3) If B_0 is a basis of $\langle A, cl^A\rangle$ and B_1 is a basis of $\langle X, cl_A\rangle$, then $B_0 \cap B_1 = \emptyset$ and $B_0 \cup B_1$ is a basis of $\langle X, cl\rangle$.

Problem 48 (8 points)**.** Prove Lemma 5.6.4.: Let T be an $\mathcal{L}$-theory, let $\mathcal{M} = \langle M, \ldots\rangle$ be a model of T, let $a_0, \ldots, a_n \in M$ and let $\mathcal{L}_*$ be the first-order language extending $\mathcal{L}$ by constant symbols $\dot{a}_0, \ldots, \dot{a}_n$. Set

$$T_* \ = \ T \ \cup \ \{\psi(\dot{a}_0, \ldots, \dot{a}_n) \mid \psi \in tp^{\mathcal{M}}(a_0, \ldots, a_n)\}.$$

Then the following statements hold:

(1) T_* is complete.
(2) Given an infinite cardinal κ, the theory T is κ-stable if and only if the theory T_* is κ-stable.
(3) If $\varphi(v_0, \ldots, v_{n+1})$ is an $\mathcal{L}$-formula with the property that there is an elementary extension $\mathcal{N}$ of M such that $\mathcal{M}$, $\mathcal{N}$ and the $\mathcal{L}_M$-formula $\varphi(v, \dot{a}_0, \ldots, \dot{a}_n)$ witness that T has a Vaughtian pair, then the canonical $\mathcal{L}_*$-expansions of $\mathcal{M}$ and $\mathcal{N}$ together with the $\mathcal{L}_*$-formula $\varphi(v, \dot{a}_0, \ldots, \dot{a}_n)$ witness that T_* has a Vaughtian pair.
(4) T has a Vaughtian pair if and only if T_* has a Vaughtian pair.

Problem 49 (4 points)**.** Prove Proposition 5.6.6.: Let $\varphi(v)$ be an $\mathcal{L}$-formula and let $\mathcal{M}$ be an $\mathcal{L}$-structure such that the following statements hold:

(1) $\varphi(\mathcal{M}) \neq \emptyset$.
(2) $\mathcal{M} \models \varphi(\dot{c})$ for every constant symbol $\dot{c}$ in $\mathcal{L}$.
(3) $\mathcal{M} \models \forall x_0, \ldots, x_n \ [(\varphi(x_0) \wedge \ldots \wedge \varphi(x_n)) \longrightarrow \varphi(f(x_0, \ldots, x_n))]$ for every $(n+1)$-ary function symbol $\dot{f}$ in $\mathcal{L}$.

Then there is a unique substructure $\mathcal{M}_\varphi$ of $\mathcal{M}$ with domain $\varphi(\mathcal{M})$ and

$$\mathcal{M}_\varphi \models \psi(a_0, \ldots, a_{n-1}) \iff \mathcal{M} \models \psi^{\varphi(v)}(a_0, \ldots, a_{n-1})$$

holds for every $\mathcal{L}$-formula $\psi(v_0, \ldots, v_{n-1})$ in which the variable v does not appear and all $a_0, \ldots, a_{n-1} \in \varphi(\mathcal{M})$.

Problem 50 (4 points)**.** Prove Lemma 5.6.7.: Let κ be an infinite cardinal, let $\mathcal{M} = \langle M, \ldots\rangle$ be a κ-saturated $\mathcal{L}$-structure and let $\varphi(v)$ be an $\mathcal{L}_M$-formula. If $\varphi(\mathcal{M})$ has cardinality less than κ, then $\varphi(\mathcal{M})$ is finite.

Please submit your solutions in the lecture on January 18.