PROBLEM SET 10 - MODEL THEORY

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Problem 41 (4 points)**.** Let κ be an infinite cardinal and let T be a complete theory with infinite models that is κ-stable. Show that for every model $\mathcal{M} = \langle M, \ldots \rangle$ of T with $|M| \geq \kappa$ and every subset A of M with $|A| \leq \kappa$, there is an elementary submodel $\mathcal{M}_0 = \langle M_0, \ldots \rangle$ of $\mathcal{M}$ such that $A \subseteq M_0$ and the sets

- M_0,
- $\{\dot{c}^{\mathcal{M}_0} \mid \dot{c} \textit{ constant symbol in } \mathcal{L}\}$,
- $\{\dot{f}^{\mathcal{M}_0} \mid \dot{f} \textit{ constant symbol in } \mathcal{L}\}$,
- $\{\dot{P}^{\mathcal{M}_0} \mid \dot{P} \textit{ predicate symbol in } \mathcal{L}\}$

all have cardinality at most κ (*Hint: use Problem 38*).

Problem 42 (4 points)**.** Let $\mathcal{M} = \langle M, \ldots \rangle$ be an $\mathcal{L}$-structure with the property that for some natural number $N > 0$ and all $A \subseteq M$, the isolated types are dense in $S_N^{\mathcal{M}}(A)$. Show that for all natural numbers $n > 0$ and all $A \subseteq M$, the isolated types are dense in $S_n^{\mathcal{M}}(A)$.

Given a set X, we let $\mathcal{P}(X)$ denote the *power set* of X, i.e. the set consisting of all subsets of X. A pair $\langle X, cl \rangle$ is a *pregeometry* if $cl : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ is a function and the following statements hold for all $A \subseteq X$ and $a, b \in X$:

(1) (Reflexivity) $A \subseteq cl(A)$.
(2) (Finite character) $cl(A) = \bigcup\{cl(B) \mid B \subseteq A \textit{ finite}\}$.
(3) (Transitivity) $cl(cl(A)) = cl(A)$.
(4) (Exchange) If $a \in cl(A \cup \{b\}) \setminus cl(A)$, then $b \in cl(A \cup \{a\})$.

Problem 43 (2 points)**.** Let K be a field, let V be a K-vector space and let $cl : \mathcal{P}(V) \longrightarrow \mathcal{P}(V)$ denote the linear closure operation. Show that $\langle V, cl \rangle$ is a pregeometry.

Problem 44 (4 points)**.** Let $\mathcal{M} = \langle M, \ldots \rangle$ be an $\mathcal{L}$-structure. Show that, if $acl : \mathcal{P}(M) \longrightarrow \mathcal{P}(M)$ denotes the function that sends a subset of M to its algebraic closure (see Definition 3.2.8), then the pair $\langle M, acl \rangle$ satisfies the above properties (1)–(3).

Let $\langle X, cl \rangle$ be a pregeometry.

(1) A subset $A \subseteq X$ is *independent* if $a \notin cl(A \setminus \{a\})$ holds for all $a \in A$.
(2) A subset $A \subseteq X$ is a *generating set* if $X = cl(A)$.
(3) A subset $A \subseteq X$ is a *basis* if A is an independent generating set.

Problem 45 (4 points)**.** Let $\langle X, cl\rangle$ be a pregeometry. Prove the following statements:

(1) If $C \subseteq X$ is a generating set and $A \subseteq C$ is independent, then there is a basis B of $\langle X, cl\rangle$ with $A \subseteq B \subseteq C$.
(2) Two bases of $\langle X, cl\rangle$ have the same cardinality.

The above statements show that for every pregeometry $\langle X, cl\rangle$, there is a unique cardinal $dim(X, cl)$ with the property that every basis of $\langle X, cl\rangle$ has cardinality $dim(X, cl)$.

Problem 46 (2 points)**.** Let K be a field, let V be a K-vector space and let $cl : \mathcal{P}(V) \longrightarrow \mathcal{P}(V)$ denote the linear closure operation. Show that $dim(V, cl)$ is equal to the K-dimension of V.

Happy Holidays!

Please submit your solutions in the lecture on January 11.