PROBLEM SET 9 - MODEL THEORY

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Problem 38 (5 points)**.** Let κ be an infinite cardinal and let T be a complete theory with infinite models. If T is κ-stable, then there is a set Γ of $\mathcal{L}$-formulas of cardinality κ with the property that for every $\mathcal{L}$-formula $\varphi(v_0, \dots, v_{n-1})$, there is an $\mathcal{L}$-formula $\psi(v_0, \dots, v_{n-1})$ in Γ with

$$T \vdash \forall x_0, \dots, x_{n-1} \ [\varphi(x_0, \dots, x_{n-1}) \longleftrightarrow \psi(x_0, \dots, x_{n-1})].$$

Problem 39 (10 points)**.** An $\mathcal{L}$-theory T has the *order property* if there is an $\mathcal{L}$-formula $\varphi(v_0, \dots, v_{2n-1})$, a model $\mathcal{M} = \langle M, \dots \rangle$ of T and a sequence

$$\langle \langle a_0^i, \dots, a_{n-1}^i \rangle \in M^n \mid i \in \mathbb{N} \rangle$$

such that

$$i < j \iff \mathcal{M} \models \varphi(a_0^i, \dots, a_{n-1}^i, a_0^j, \dots, a_{n-1}^j)$$

holds for all $i, j \in \mathbb{N}$.

(1) If T is a complete theory with the order property and I is an infinite linear order, then there is an $\mathcal{L}$-formula $\varphi(v_0, \dots, v_{2n-1})$, a model $\mathcal{M} = \langle M, \dots \rangle$ of T of cardinality $|I| + |\mathcal{L}|$ and a sequence $\langle \langle a_0^i, \dots, a_{n-1}^i \rangle \in M^n \mid i \in I \rangle$ such that

$$i <_I j \iff \mathcal{M} \models \varphi(a_0^i, \dots, a_{n-1}^i, a_0^j, \dots, a_{n-1}^j)$$

holds for all $i, j \in I$.

(2) Show that for every infinite cardinal κ, there is a linear order of cardinality greater than κ that contains a dense subset of cardinality at most κ.

(3) If T is a complete theory with the order property, then T is not κ-stable for any infinite cardinal κ.

Problem 40 (5 points)**.** Complete the proof of Lemma 5.3.8 that shows that every first-order language has a Skolem theory.

Please submit your solutions in the lecture on December 21.