PROBLEM SET 7 - MODEL THEORY

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Problem 26. (5 points) Show that for every countable model $\mathcal{M} = (M, \dots)$ of $T = \mathrm{Th}(\mathbb{N}, 0, +, \cdot, <)$, there is a countable elementary *end extension* $\mathcal{N} = (N, \dots)$ of $\mathcal{M}$, i.e. $\mathcal{N} \models m < n$ for all $m \in M$ and $n \in N \setminus M$.

To do this, consider the *elementary diagram* $\mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$ of $\mathcal{M}$, i.e. the set of $\mathcal{L}_M$-sentences that hold in $\mathcal{M}$. Let $T = \mathrm{Diag}_{\mathrm{el}}(\mathcal{M}) \cup \{c > m \mid m \in M\}$, where c is a new constant symbol. Then show that the set $\Sigma_n(v) = \{v < n \wedge v \neq m \mid m \in M\}$ is not isolated by any $\mathcal{L}_M \cup \{c\}$-formula $\theta(v, c)$ for any $n \in N \setminus M$ by proving the following statements.

(a) $\mathcal{M} \models \forall x\ \exists y > x\ \exists v < n\ \theta(v, y)$.
(b) There is some $m < n$ such that $\mathcal{M} \models \forall x\ \exists y > x\ \theta(m, y)$.
Hint: the pigeonhole principle holds in $(\mathbb{N}, 0, +, \cdot, <)$.
(c) $\theta(m, c)$ is consistent with T.

Problem 27. (5 points) Let $\mathcal{L} = \{<, c_0, c_1, \dots\}$ and T the $\mathcal{L}$-theory that consists of the axioms for dense linear orders together with the axioms $\neg\exists x\ \forall y\ (x = y \vee x < y)$, $\neg\exists x\ \forall y\ (x = y \vee y < x)$ and $c_i < c_j$ for all $i \neq j$ in $\mathbb{N}$. Prove the following statements.

(a) T has quantifier elimination and is complete.
(b) T has exactly three countable models up to isomorphism.

Problem 28. (4 points) For the following structures $\mathbb{M} = (M, \dots)$, determine for any pair of elements $a, b \in M$ if they realize the same or different 1-types. Are there 1-types with respect to $\mathrm{Th}(\mathbb{M})$ that are not realized?

(a) $(\mathbb{Z}, +)$
(b) $(\mathbb{Z}, \cdot)$

Problem 29. (3 points) Decide whether the expansion $(\mathbb{R}, <, Q)$ of $(\mathbb{R}, <)$ with an additional predicate for the set of rational numbers has a prime model.

Problem 30. (3 points) Show that there is no $\aleph_0$-categorical $\mathcal{L}_R$-theory that contains the axioms for fields.

Problem 31. *(2 extra points) Find a countable theory with exactly two models of size $\aleph_0$ up to isomorphism.

Problem 32. *(2 extra points) For each $n \geq 1$, find a countable complete theory T such that there is a unique n-type, but $S_{n+1}(T)$ is uncountable.

Please submit your solutions in the lecture on December 8.