PROBLEM SET 5 - MODEL THEORY

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This time there are personalized problems for mini-presentations of 10-20 minutes.

(1) Show that the compactness theorem is equivalent to the ultrafilter lemma in ZF, using for example the proof of Theorem 5.15 in Halbeisen's book Combinatorial set theory. Moreover show that the Löwenheim-Skolem theorem for countable elementary submodels is equivalent to the axiom of dependent choices DC in ZF (Stefan, Thorben)

(2) Suppose that U is a non-principal ultrafilter on $\mathbb{N}$. Show that the field $\prod_{n\in\mathbb{N}}^{U}\mathbb{R}$ is an elementry extension of the field $\mathbb{R}$ of reals with an uncountable transcendence base over $\mathbb{R}$. Moreover show that for any function $f\colon \mathbb{R}\to\mathbb{R}$ with $f(0)=0$ and any elementary extension $(\mathbb{R}^*,0,1,+,-,\cdot,f^*)$ of $(\mathbb{R},0,1,+,-,\cdot,f)$ with non-zero infinitesimals (i.e. numbers x with $-\frac{1}{n}<x<\frac{1}{n}$ for all $n\in\mathbb{N}$), f is continuous at 0 if and only if f^* maps infinitesimals to infinitesimals (Joao)

(3) Determine which of the following fields are isomorphic to $\mathbb{C}$, where U is a non-principal ultrafilter on the set P of primes or $\mathbb{N}$, respectively: $\prod_{p\in P}^{U}\mathbb{F}_p$, $\prod_{p\in P}^{U}\bar{\mathbb{F}}_p$, $\prod_{n\in\mathbb{N}}^{U}\mathbb{C}$; these denote the ultrapowers by U (Nikola)

(4) Give a proof of the Tarski-Seidenberg theorem following Section 3.3.5 of the book by Tent and Ziegler from the results in algebra that are stated there (Pier Giorgio)

(5) Show that the theory of discrete linear orderings without end points with a successor and a predecessor function has quantifier elimination and is complete (Oliver)

(6) Let $\mathcal{L}$ be the language with a unary relation symbol P_s for every finite sequence $s\in 2^{<\omega}$ and T the $\mathcal{L}$-theory of binary trees that consists of the axioms

$$\forall x P_{\emptyset}(x)\wedge\exists x P_s(x)\wedge\forall x((P_{s0}(x)\vee P_{s1}(x))\leftrightarrow P_s(x))\wedge\forall x\neg(P_{s0}(x)\wedge P_{s1}(x))$$

for all $s\in 2^{<\omega}$. Show that T has quantifier elimination and is complete but has no isolated types and no prime models (Franziska, Joshua)

(7) Show that the theories of the structures $(\mathbb{N},S)$ and $(\mathbb{N},+)$ do not satisfy quantifier elimination, where S is the successor function (Sebastian)

(8) Show that the class $\mathcal{K}$ of finitely generated torsion-free abelian groups is an amalgamation class whose Fraisse limit is a direct sum of countably many copies of $(\mathbb{Q},+)$ (Chiara)

(9) Show that the class $\mathcal{K}$ of finite Boolean algebras is an amalgamation class whose Fraisse limit is the countable atomless Boolean algebra (Timo)

Please present your solutions in the tutorial on November 27.