PROBLEM SET 3 - MODEL THEORY

Philipp Schlicht and Philipp Lücke
Mathematisches Institut, Universität Bonn, Winter 2017-18

Problem 11. (3 points) Show that any two finite elementarily equivalent $\mathcal{L}$-structures $\mathbb{M}$ and $\mathbb{N}$ in a finite language $\mathcal{L}$ are isomorphic.

Problem 12. (8 points) Decide if the following classes of $\mathcal{L}$-structures for $\mathcal{L} = \{\cdot, 1\}$ are axiomatizable by giving an axiomatization or proving that it is not by using either the compactness theorem or ultrapowers. Let G^n

(a) Let $\mathcal{K}$ be the class of groups G with $\bigcap_{n\in\mathbb{N}} G^n = \{1\}$, where $G^n = \{g^n \mid n \in \mathbb{N}\}$ denotes the set of n^{th} powers of elements of G.
(b) Let $\mathcal{K}$ be the class of torsion free groups.
(c) Let $\mathcal{K}$ be the class of torsion groups, i.e. where for every $g \in G$ there is some $n > 0$ with $g^n = 1$.
(d) Let $\mathcal{K}$ be the class of free groups. *Hint: use the fact that no free group is abelian unless it is isomorphic to* $(\mathbb{Z}, 0, +, -)$.

Problem 13. (4 points) Prove the compactness theorem by using ultrapowers: every finitely satisfiable theory T is satisfiable. *Hint: take as index set the set I of all finite subsets of T and choose an ultrafilter U on I that contains for every $\varphi \in T$ the set $I_\varphi = \{S \in I \mid \varphi \in S\}$; you can use the fact that every filter can be extended to an ultrafilter.*

Problem 14. (5 points) An $\mathcal{L}$-theory T is called *universal* if it has the same models as the set $T_\forall = \{\varphi \in \mathrm{Sent}_\mathcal{L} \mid \varphi \text{ is universal and } T \models \varphi\}$ of its universal consequences. A theory T is called *downwards absolute* if for any substructure of any model $\mathbb{M}$ of T is also a model of T. Moreover, the *atomic diagram* $\mathrm{Diag}(\mathbb{M})$ of an $\mathcal{L}$-structure $\mathbb{M} = (M, \dots)$ is the set of atomic $\mathcal{L}_M$-sentences that hold in $\mathbb{M}$. Prove the following statements.

(a) Every universal theory is downwards absolute.
(b) If T is downwards absolute, then for any model $\mathbb{M} = (M, \dots)$ of $T_\forall$, the theory $T \cup \mathrm{Diag}(\mathbb{M})$ is finitely satisfiable.
(c) Every downwards absolute theory is universal.

Please submit your solutions in the lecture on November 9.