PROBLEM SET 2 - MODEL THEORY

Philipp Schlicht and Philipp Lücke
Mathematisches Institut, Universität Bonn, Winter 2017-18

Problem 6. (2 points) Show that no finite structure $\mathbb{M}$ has any proper elementary substructures or extensions.

Problem 7. (4 points) If $\mathbb{M} = (M, \dots)$ and $\mathbb{N} = (N, \dots)$ are structures and $h\colon M \to N$ is a function, the following conditions are equivalent.

(a) h is an embedding.

(b) For every atomic formula $\varphi(x_0, \dots, x_{n-1})$ and all $a_0, \dots, a_{n-1} \in M$, we have $\mathbb{M} \models \varphi(a_0, \dots, a_{n-1})$ if and only if $\mathbb{N} \models \varphi(h(a_0)), \dots h(a_{n-1}))$.

(c) For every quantifier-free formula $\varphi(x_0, \dots, x_{n-1})$ and all $a_0, \dots, a_{n-1} \in M$, we have $\mathbb{M} \models \varphi(a_0, \dots, a_{n-1})$ if and only if $\mathbb{N} \models \varphi(h(a_0)), \dots h(a_{n-1}))$.

Problem 8. (3 points) We have $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ when these are viewed as structures in $\mathcal{L}_R = \{0, 1, +, -, \cdot\}$. Show that none of these inclusions is elementary.

Problem 9. (6 points) Let T be an $\mathcal{L}_R$-theory that contains the axioms of the theory of fields. Prove the following statements.

(a) If T has models of arbitrarily large characteristic, then it has a model of characteristic 0.

(b) The theory of fields of characteristic 0 is not finitely axiomatizable.

Problem 10. (5 points) Let $\mathbb{G}_n = (G_n, E_n)$ be the (undirected) graph that consists of exactly one cycle with n elements, U a non-principal ultrafilter on $\mathbb{N}$ and $\mathbb{G} = (G, E) = \prod_{n \in \mathbb{N}}^U \mathbb{G}_n$ the ultraproduct of these graphs with respect to U.

(a) Show that $\mathbb{G}$ is a non-connected graph.

(b) Determine the structure of all connected components of $\mathbb{G}$.

Please submit your solutions in the lecture on November 2.