PROBLEM SET 1 - MODEL THEORY

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Problem 1. (4 points) Show that the completeness and compactness theorems follow from the fact that every syntactically consistent set of $\mathcal{L}$-formulas has a model with an assignment of variables for any language $\mathcal{L}$.

(a) (Completeness) If T is a set of $\mathcal{L}$-formulas and φ is an $\mathcal{L}$-formula, then $T \vdash \varphi$ if and only if $T \models \varphi$.

(b) (Compactness) Every finitely satisfiable $\mathcal{L}$-theory is satisfiable.

Problem 2. (3 points) Suppose that $\mathbb{N} = (N, \dots)$ is an $\mathcal{L}$-structure, $a_0, \dots, a_{n-1} \in N$ and $M = \{t^{\mathbb{N}}(a_0, \dots, a_{n-1}) \mid t(x_0, ..., x_{n-1}) \text{ is an } \mathcal{L}\text{-term}\}$. Prove that M is the least subset of N that is the domain of a substructure of $\mathbb{N}$ containing $a_0, \dots, a_{n-1}$.

Problem 3. (3 points) Prove that $(\mathbb{Q}, <) \prec (\mathbb{R}, <)$. *Hint: show by induction that for all formulas $\varphi(x_0, \dots, x_{n-1})$ in the language $\mathcal{L} = \{<\}$ and all $a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1} \in \mathbb{R}$ with $(\mathbb{R}, <) \models \bigwedge_{i,j<n}(a_i < a_j \leftrightarrow b_i < b_j)$, we have $(\mathbb{R}, <) \models \varphi(a_0, \dots, a_{n-1}) \leftrightarrow \varphi(b_0, \dots, b_{n-1})$.*

Problem 4. (4 points)

(a) Show that any two isomorphic $\mathcal{L}$-structures $\mathbb{M}$ and $\mathbb{N}$ are elementarily equivalent.

(b) Show that $(n\mathbb{Z}, +) \equiv (\mathbb{Z}, +)$ for all $n \geq 1$.

(c) Is $(n\mathbb{Z}, +)$ an elementary substructure of $(\mathbb{Z}, +)$ for $n > 1$?

Problem 5. (6 points) If X is a set, let $X^{<\omega}$ denote the set of finite sequences with values in X and $[X]^{<\omega}$ the set of finite subsets of X. Assuming that κ is an infinite cardinal, prove the following statements by using the fact that $|X \times Y| = \max\{\lambda, \mu\}$ for all infinite cardinals λ, μ and sets X and Y with $|X| = \lambda$ and $|Y| = \mu$.

(a) If $\langle X_i \mid i \in I \rangle$ is a sequence with $|X_i| \leq \kappa$ for all $i \in I$ and $|I| \leq \kappa$, then $|\bigcup_{i \in I} X_i| \leq \kappa$.

(b) If $|X| = \kappa$, then $|X^{<\omega}| = \kappa$.

(c) If $|X| = \kappa$, then $|[X]^{<\omega}| = \kappa$.

Please submit your solutions in the lecture on October 26.