Models of Set Theory I – Summer 2017

Prof. Peter Koepke, Dr. Philipp Lücke – Problem Sheet 3

Problem 9 [2 points] Verify the following, working in ZF.

1. $V = \mathrm{HOD}$ if and only if there is a wellorder of V of order-type Ord that is definable without parameters.

2. If there is a set X without a wellordering, then there is some $x \in X$ so that $x \notin \mathrm{HOD}$.

Problem 10 [4 points] Working in ZF, show that the following are equivalent.

1. $V = \mathrm{OD}$

2. $V = \mathrm{HOD}$

3. OD is transitive

4. Extensionality is true in OD

Problem 11 [8 points] Work in ZF. If $r \subseteq \omega$, we generalize OD to $\mathrm{OD}[r]$, and we generalize HOD to $\mathrm{HOD}[r]$, by allowing for the parameter r in their definitions, that is $\mathrm{OD}[r]$ is the collection of sets that are definable from ordinals together with the additional parameter r, and we let $x \in \mathrm{HOD}[r]$ if and only if $TC(\{x\}) \subseteq \mathrm{OD}[r]$.

1. Provide a definition of $\mathrm{OD}[r]$ by a formula in the language of set theory which corresponds to the above informal definition, such that $r \in \mathrm{HOD}[r]$, and verify this latter property.

2. Show that $\mathrm{HOD}[r]$ is transitive.

3. Show that $\mathrm{OD}[r]$ has a wellorder of order-type Ord which is definable using r as parameter.

4. Show that ZFC holds in $\mathrm{HOD}[r]$.

Problem 12 [6 points] Work in ZFC. Let κ be a measurable cardinal, and let U be a κ-complete filter on κ. Let P be the collection of all p of the form $p = \langle s, A \rangle$ where s is a finite set of ordinals less than κ, $A \in U$, and $\max s < \min A$. Given $\langle s, A \rangle$ and $\langle t, B \rangle$ both in P, we write $\langle t, B \rangle \leq \langle s, A \rangle$ if the following conditions hold true:

- $t \cap (\max(s) + 1) = s$,
- $B \subseteq A$ and
- $t \setminus s \subseteq A$.

Show that

1. $\mathbb{P} = \langle P, \leq \rangle$ is a partial order.
2. $\mathbb{P}$ has the κ^+-cc.