Set theory - Winter semester 2016-17

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Problem 37 (4 points)**.** Suppose that κ is an uncountable regular cardinal and $f\colon \kappa \to \kappa$ is a function. Then there is a stationary subset S of κ such that $f\restriction S$ is constant or $f\restriction S$ is injective.

Problem 38 (4 points)**.** Suppose that $\kappa > \omega$ is a regular cardinal. A function $f\colon \kappa \to \kappa$ is called *normal* if it is strictly monotone and continuous. Prove the following statements.

(1) If $f\colon \kappa \to \kappa$ is normal, then $f[\kappa]$ is club in κ.

(2) If C is club in κ, then the strictly monotone enumeration $f\colon \kappa \to \kappa$ of C is normal.

Problem 39 (6 points)**.** Suppose that $\kappa > \omega$ is a regular cardinal. For every function $f\colon \kappa \to \kappa$, the set C_f of *closure points* of f is defined as

$$C_f = \{\alpha < \kappa \mid f[\alpha] \subseteq \alpha,\ \alpha > 0\}.$$

Prove the following statements.

(1) (a) If $f\colon \kappa \to \kappa$ is a function, then C_f is club in κ.

(b) If C is club in κ, then there is a function $f\colon \kappa \to \kappa$ with $C_f \subseteq C$.

(2) If $f\colon \kappa \to \kappa$ is normal, then the set

$$\mathrm{Fix}(f) = \{\alpha < \kappa \mid f(\alpha) = \alpha\}$$

of fixed points of f is club in κ.

Problem 40 (8 points)**.** Suppose that κ is a singular cardinal of uncountable cofinality and $\langle \kappa_\alpha \mid \alpha < \mathrm{cof}(\kappa)\rangle$ is a strictly increasing continuous cofinal sequence of cardinals below κ. Prove the following statements.

(1) If $\mathcal{F} \subseteq \prod_{\alpha<\mathrm{cof}(\kappa)} A_\alpha$ is almost disjoint and $\mathrm{card}(A_\alpha) \leq \kappa_\alpha^{++}$ for all $\alpha < \mathrm{cof}(\kappa)$, then $\mathrm{card}(\mathcal{F}) \leq \kappa^{++}$.

(2) If $2^\mu \leq \mu^{++}$ for all infinite cardinals $\mu < \kappa$, then $2^\kappa \leq \kappa^{++}$.

(Hint: adapt the proof of Lemma 156 from the lecture.)

Due Friday, December 23, before the lecture.