Set theory - Winter semester 2016-17

Problems	Prof. Peter Koepke
Series 12	Dr. Philipp Schlicht

Problem 45 (2 points)**.** Suppose that U is a κ-complete ultrafilter on a cardinal κ and $f{:}\,\kappa \to \alpha$ is a function for some $\alpha < \kappa$. Show that there is a set $A \in U$ such that $f{\upharpoonright}A$ is constant.

Problem 46 (2 points)**.** Suppose that μ is a measure on a set X. Suppose that $\langle X_i \mid i < \alpha\rangle$ is a sequence of disjoint subsets of X such that $\mu(X_i) > 0$ for each $i < \alpha$. Show that α is countable.

Problem 47 (6 points)**.**

(1) Suppose that for each $\xi < \omega_1$, $f_\xi{:}\,\omega \to \omega_1$ is a function with $\xi \subseteq \operatorname{ran}(f)$. We define

$$A_{\alpha,n} = \{\xi < \omega_1 \mid f_\xi(n) = \alpha\}$$

for all $n \in \omega$ and $\alpha < \omega_1$. Prove the following properties.

   (a) If $n \in \omega$ and $\alpha < \beta < \omega_1$, then $A_{\alpha,n} \cap A_{\beta,n} = \varnothing$.

   (b) For each $\alpha < \omega_1$, the set $\omega_1 \setminus \bigcup_{n\in\omega} A_{\alpha,n}$ is at most countable.

(2) Prove that there is no nontrivial measure on ω_1 *(Hint: use the sets $A_{\alpha,n}$ and Problem 46).*

Problem 48 (8 points)**.** Suppose that μ is a nontrivial measure on $P(X)$ such that $\mu(X) = 1$ and every subset Y of X with $\mu(Y) > 0$ splits, i.e. there are disjoint subsets Y_0 and Y_1 of Y with $\mu(Y_0) > 0$ and $\mu(Y_1) > 0$.

(1) Show that for every subset Y of X and every $\epsilon > 0$, there is some n and a partition $\langle Y_i \mid i < n\rangle$ of Y such that $\mu(Y_i) < \epsilon$ for all $i < n$. *(Hint: use the result from the lecture about obtaining a subset with measure between $\frac{1}{3}$ and $\frac{2}{3}$.)*

(2) Show that for every $r \in [0,1]$, there is a subset Y of X with $\mu(Y) \le r$ and $\mu(Y) - r < \epsilon$.

(3) Show that for every $r \in [0,1]$, there is a subset Y of X with $\mu(Y) = r$.

Due Friday, January 27, 14:45-15:00, Plückerraum (opposite to the Hausdorffraum 1.012, Mathematik-Zentrum), in the folders at the windows.