Set theory - Winter semester 2016-17

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Problem 41 (4 points)**.** Suppose that the following strong form of the *singular cardinal hypothesis* SCH holds: for every singular cardinal λ with $2^{\mathrm{cof}(\lambda)} < \lambda$, $2^\lambda = \lambda^+$ holds. Determine 2^κ from $2^{<\kappa}$ for all singular cardinals κ.

Problem 42 (6 points)**.** Suppose that κ is an uncountable limit cardinal. Prove that the following sets have the same size.

(1) $\kappa^{\mathrm{cof}(\kappa)}$.
(2) The set of all cofinal functions $f\colon \mathrm{cof}(\kappa) \to \kappa$.
(3) The set of all strictly increasing cofinal functions $f\colon \mathrm{cof}(\kappa) \to \kappa$.
(4) $\prod_{i<\mathrm{cof}(\kappa)} g(i)$, where $g\colon \mathrm{cof}(\kappa) \to \mathrm{Card}\cap\kappa$ is any strictly increasing cofinal function.

Problem 43 (6 points)**.** Suppose that S is a stationary subset of ω_1. Prove for all ordinals $\beta < \omega_1$ by induction that the set S_β of all $\gamma \in S$ with the following property is stationary:

For all $\alpha < \gamma$, there is a closed subset C of S with $\min(C) \geq \alpha$, $\mathrm{otp}(C) = \beta + 1$, *and* $\sup(C) = \gamma$.

(Hint: work with the club $\bigcap_{\alpha<\beta} \lim(S_\alpha)$, where $\lim(A)$ denotes the set of limit points below ω_1 of a set $A \subseteq \omega_1$.)

Problem 44 (4 points)**.** Suppose that $\kappa \leq \lambda$ are regular uncountable cardinals and let $P_\kappa(\lambda) = \{x \subseteq \lambda \mid \mathrm{card}(x) < \kappa\}$.

(a) A subset F of $P_\kappa(\lambda)$ is a *filter on* $P_\kappa(\lambda)$ if it satisfies the conditions in Definition 76.
(b) If F is a filter on $P_\kappa(\lambda)$, the set F^+ of F*-positive sets* is defined as $F^+ = \{x \in P_\kappa(\lambda) \mid \forall y \in F\ x \cap y \neq \emptyset\}$.
(c) A function $f\colon D \to \lambda$ with $D \subseteq P_\kappa(\lambda)$ is called *regressive* if $f(x) \in x$ for all $x \in D$.

Suppose that for every sequence $\langle X_i \mid i < \lambda \rangle$ of elements of F,

$$\triangle_{i<\lambda} X_i := \{x \in P_\kappa(\lambda) \mid x \in \bigcap_{\alpha \in x} X_\alpha)\} \in F.$$

Prove that for every $D \in F^+$ and regressive function $f\colon D \to \lambda$, there is a subset $E \in F^+$ of D such that f is constant on E.

Due Friday, January 20, before the lecture.