Set theory - Winter semester 2016-17

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Problem 33 (6 points)**.** Let S_α denote the set of finite strictly decreasing sequences of ordinals strictly below $\alpha \in \mathrm{Ord}$ and

$$S = \bigcup_{\alpha \in \mathrm{Ord}} S_\alpha.$$

Let $\vec{s} = (s_0, \ldots, s_m) <^* \vec{t} = (t_0, \ldots, t_n)$ for $\vec{s}, \vec{t} \in S$ if

(i) $n < m$ and $s_i = t_i$ for all $i < n$ or

(ii) there is some $i \leq \min(\{m, n\})$ with $s(i) \neq t(i)$, and for the least such i, $s(i) < t(i)$.

Prove the following statements.

(a) $<^*$ is a wellfounded relation on S *(hint: you can use Problem 15).*

(b) $(S_{\omega+\omega}, <^*)$ is isomorphic to $(\omega \cdot \omega, <)$, where $\omega + \omega$ denotes the ordinal sum and $\omega \cdot \omega$ denotes the ordinal product.

Problem 34 (4 points)**.** Prove the following statements.

(a) If λ is a limit cardinal, then there is a cofinal function $f \colon \mathrm{cof}(\lambda) \to \mathrm{Card} \cap \lambda$.

(b) If λ is an infinite cardinal, then $\mathrm{cof}(\lambda)$ is equal to the least ordinal γ such that there is a sequence $\langle \kappa_i \mid i < \gamma \rangle$ such that $\kappa_i \in \mathrm{Card} \cap \lambda$ for all $i < \gamma$ and $\sum_{i<\gamma} \kappa_i = \lambda$.

Problem 35 (4 points)**.** (a) Show that for every transitive set x, $(x, \in)$ satisfies the Axiom of Extensionality.

(b) Suppose that κ is (strongly) inaccessible. Show that V_κ is closed under the following operations.

(i) Power sets, pairs and unions.

(ii) For every formula $\varphi(x, y, z)$, the map f_φ sending an ordered pair (y, z) to the set

$$f_\varphi(y, z) = \{x \in y \mid \varphi(x, y, z)\}.$$

(iii) For every formula $\varphi(a, b, c, d)$, the map g_φ sending an ordered pair (c, d) to the set

$$g_\varphi(c, d) = \{(a, b) \in V_\kappa \mid a \in c \wedge \varphi(a, b, c, d)\}$$

if this is a function, and to $g_\varphi(c, d) = 0$ otherwise.

Problem 36 (6 points)**.** Suppose that the GCH holds. Then for $\kappa, \lambda \in \mathrm{Card}$,

(a) For $\lambda < \mathrm{cof}(\kappa), \kappa^\lambda = \kappa$.
(b) For $\mathrm{cof}(\kappa) \leq \lambda \leq \kappa$, $\kappa^\lambda = \kappa^+$.
(c) For $\lambda > \kappa$, $\kappa^\lambda = \lambda^+$.

Due Friday, December 16, before the lecture.