Models of Set Theory II - Winter 2015/2016

Prof. Dr. Peter Koepke, Regula Krapf — Problem sheet 7

Problem 27 (4 points)**.** Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a model of GB. Show that the following statements are equivalent:

(1) $\mathbb{M} \models \mathsf{GBC}$.
(2) $\mathcal{C}$ contains a global well-order of M.
(3) $\mathcal{C}$ contains a bijection between M and Ord.

Problem 28 (8 points)**.** Let $\mathbb{P}$ be a class forcing for $\mathbb{M} = \langle M, \mathcal{C}\rangle \models \mathsf{GB} + \mathsf{AC}$. For $A \in M$ with $A \subseteq \mathbb{P}$ and $p \in \mathbb{P}$ we say that p *decides* A, if for every $a \in A$, either $p \leq_{\mathbb{P}} a$ or $p \perp_{\mathbb{P}} a$. We say that $\mathbb{P}$ has the *set decision property*, if for every $p \in \mathbb{P}$ and for every $A \subseteq \mathbb{P}$ with $A \in M$ there is $q \leq_{\mathbb{P}} p$ which decides A.

(a) Suppose that $\mathbb{P}$ has the set decision property. Show that $\mathbb{P}$ does not add any new sets, i.e. if G is $\mathbb{M}$-generic for $\mathbb{P}$ then $M[G] = M$.
(b) Show that every class forcing with the set decision property satisfies the forcing theorem for the atomic formulae "$v_0 \in v_1$" and "$v_0 = v_1$". *(Note that from this it follows that the forcing theorem holds for all formulae.)*
(c) Prove that every $<$ Ord-closed class forcing has the set decision property.
(d) Conclude that every $<$ Ord-closed class forcing preserves all axioms of $\mathsf{GB} + \mathsf{AC}$.

Hint for (a): For $p \in \mathbb{P}$ and $\sigma \in M^{\mathbb{P}}$, consider

$$\sigma^p = \{\tau^p \mid \exists q(\langle \tau, q\rangle \in \sigma \wedge p \leq_{\mathbb{P}} q)\}.$$

Problem 29 (8 points)**.** Let $\mathrm{Col}_*(\omega, \mathrm{Ord})$ denote the class forcing with conditions $p : \mathrm{dom}(p) \to \mathrm{Ord}$ with $\mathrm{dom}(p) \subseteq \omega$ finite, ordered by reverse inclusion, and let $\mathrm{Col}_*(\omega, \mathrm{Ord})$ denote the subforcing of $\mathrm{Col}(\omega, \mathrm{Ord})$ of conditions p with $\mathrm{dom}(p) \in \omega$. Prove the following statements:

(a) $\mathrm{Col}(\omega, \mathrm{Ord})$ and $\mathrm{Col}_*(\omega, \mathrm{Ord})$ are not pretame.
(b) Power set fails in every $\mathrm{Col}(\omega, \mathrm{Ord})$-generic extension.
(c) $\mathrm{Col}_*(\omega, \mathrm{Ord})$ has the set decision property.
(d) Conclude that there exists a dense embedding $\mathbb{P} \to \mathbb{Q}$ of class forcings such that $\mathbb{P}$ and $\mathbb{Q}$ are not forcing equivalent, i.e. they do not generate the same generic extensions.

Problem 30 (2 points)**.** Let $\mathbb{P}$ be a class forcing for $\mathbb{M} = \langle M, \mathcal{C} \rangle \models \mathsf{GBC}$ and suppose that for every $p \in \mathbb{P}$ there is an $\mathbb{M}$-generic filter $G \subseteq \mathbb{P}$ with $p \in G$ such that $\mathbb{M}[G]$ is a model of GBC^-. Then $\mathbb{P}$ is pretame.

Hint: Given dense classes $\langle D_i \mid i < \kappa \rangle$ consider the function which maps some $i < \kappa$ to the least rank of an element in $D_i \cap G$.

Please hand in your solutions on Monday, 21.12.2015 before the lecture.