Models of Set Theory II - Winter 2015/2016

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Problem 19 (2 points). Let $\langle\langle \mathbb{P}_\alpha, \leq_\alpha, \mathbb{1}_\alpha\rangle \mid \alpha \leq \kappa\rangle$ denote the finite support iteration of the sequence $\langle\langle \dot{\mathbb{Q}}_\alpha, \dot{\leq}_\alpha\rangle \mid \alpha < \kappa\rangle$. Let G be M-generic for $\mathbb{P}_\kappa$ and G_α, H_α be the derived generic filters for $\mathbb{P}_\alpha$ resp. $\dot{\mathbb{Q}}_\alpha^{G_\alpha}$. Show that for each $\alpha \leq \kappa$, $M[G_\alpha] = M[\langle H_\beta \mid \beta < \alpha\rangle]$, where $M[\langle H_\beta \mid \beta < \alpha\rangle]$ is the smallest model N of ZFC with $M \cup \{\langle H_\beta \mid \beta < \alpha\rangle\} \subseteq N$.

Problem 20 (6 points). Let $\mathbb{P}$ and $\mathbb{Q}$ denote forcing notions for M. A *projection* from $\mathbb{P}$ to $\mathbb{Q}$ is a map $\pi : \mathbb{P} \to \mathbb{Q}$ such that

(1) $\pi(\mathbb{1}_\mathbb{P}) = \mathbb{1}_\mathbb{Q}$
(2) If $p_0 \leq_\mathbb{P} p_1$ then $\pi(p) \leq_\mathbb{Q} \pi(q)$
(3) If $p \in \mathbb{P}$ and $q \in \mathbb{Q}$ such that $q \leq_\mathbb{Q} \pi(p)$ then there is $\bar{p} \leq_\mathbb{P} p$ with $\pi(\bar{p}) \leq_\mathbb{Q} q$.

Suppose that $\pi : \mathbb{P} \to \mathbb{Q}$ is a projection. Prove the following statements:

(a) If G is M-generic for $\mathbb{P}$ then the pointwise image of G under π generates an M-generic filter on $\mathbb{Q}$.
(b) Suppose that H is M-generic for $\mathbb{Q}$ and let $\mathbb{P}/H = \{p \in \mathbb{P} \mid \pi(p) \in H\}$ with the ordering inherited from $\mathbb{P}$. If G is $M[H]$-generic for $\mathbb{P}/H$ then G is M-generic for $\mathbb{P}$.
(c) Show that if $\langle\langle \mathbb{P}_\alpha, \leq_\alpha, \mathbb{1}_\alpha\rangle \mid \alpha \leq \kappa\rangle$ is a finite support iteration then the restriction maps $\mathbb{P}_\beta \to \mathbb{P}_\alpha$ for $\alpha < \beta \leq \kappa$ are projections.

Let $\mathbb{R}$ denote the set of reals. A set $\mathcal{I} \subseteq \mathcal{P}(\mathbb{R})$ is said to be an *ideal* on $\mathbb{R}$, if it has the following properties:

(1) if $X, Y \in \mathcal{I}$ then $X \cup Y \in \mathcal{I}$.
(2) if $X \in \mathcal{I}$ and $Y \subseteq X$ then $Y \in \mathcal{I}$.
(3) for each $x \in \mathbb{R}$, $\{x\} \in \mathcal{I}$
(4) $\mathbb{R} \notin \mathcal{I}$.

Let $\mathcal{I}$ be an ideal on $\mathbb{R}$. We define

- $\mathrm{cov}(\mathcal{I}) = \min\{\mathrm{card}(\mathcal{A}) \mid \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} = \mathbb{R}\}$
- $\mathrm{non}(\mathcal{I}) = \min\{\mathrm{card}(X) \mid X \subseteq \mathbb{R} \wedge X \notin \mathcal{I}\}$.

Let $\mathcal{N}$ denote the ideal of null sets (i.e. Lebesgue measure zero sets) and let $\mathcal{M}$ denote the ideal of meager sets.

Problem 21 (4 points)**.** Let $\mathbb{P} = \mathrm{Fn}(\aleph_1 \times \omega, 2, \aleph_0)$ denote the forcing notion for adding $\aleph_1$-many Cohen reals. Prove the following statements:

(a) If G is M-generic for $\mathbb{P}$ then $M[G]$ contains an $\aleph_1$-sequence of measure 0 sets whose union is $\mathbb{R}$.

(b) It is consistent that $\aleph_1 = \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{N}) = 2^{\aleph_0}$.

We define two more *cardinals invariants.* A family $\mathcal{F} \subseteq {}^{\omega}\omega$ is said to be *dominating*, if for every $g \in {}^{\omega}\omega$ there is $f \in \mathcal{F}$ such that $g \leq_* f$. Furthermore, $\mathcal{F}$ is said to be *unbounded*, if $\mathcal{F}$ has no upper bound with respect to $\leq_*$.

- The *dominating number* $\mathfrak{d}$ is the least size of a dominating family in ${}^{\omega}\omega$.
- The *bounding number* $\mathfrak{b}$ is the least size of an unbounded family in ${}^{\omega}\omega$.

Problem 22 (6 points)**.** Prove the following statements:

(a) $\mathfrak{b} \leq \mathrm{non}(\mathcal{M})$.

(b) $\mathfrak{b} \leq \mathrm{cf}(\mathfrak{d})$.

(c) $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{d}$.

Please hand in your solutions on Monday, 07.12.2015 before the lecture.