Models of Set Theory II - Winter 2015/2016

Prof. Dr. Peter Koepke, Regula Krapf — Problem sheet 3

Problem 11 (4 points)**.** Let $\mathbb{P}$ and $\mathbb{Q}$ denote forcing notions.

(a) Suppose that $\pi : \mathbb{P} \to \mathbb{Q}$ is a complete embedding as defined in Problem 13, Models of Set Theory I. We define a map $M^{\mathbb{P}} \to M^{\mathbb{Q}}$ by recursion on $\sigma \in M^{\mathbb{P}}$ by

$$\pi^*(\sigma) = \{\langle \pi^*(\tau), \pi(p)\rangle \mid \langle \tau, p\rangle \in \sigma\}.$$

Suppose that H is M-generic for $\mathbb{Q}$ and let $G = \pi^{-1}[H]$ denote the corresponding M-generic filter for $\mathbb{P}$. Prove that for all $\sigma \in M^{\mathbb{P}}$, $\sigma^G = \pi^*(\sigma)^H$.

(b) Show that the map $M^{\mathbb{P}} \to M^{\mathbb{P}}$ defined recursively by mapping σ to

$$\bar{\sigma} = \{\langle \bar{\tau}, \mathbb{1}_{\mathbb{P}}\rangle \mid \exists p(\langle \tau, p\rangle \in \sigma)\}$$

is not well-defined, i.e. there are $\sigma, \tau \in M^{\mathbb{P}}$ such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}^{M} \sigma = \tau$ and $\mathbb{1}_{\mathbb{P}} \nVdash_{\mathbb{P}}^{M} \bar{\sigma} = \bar{\tau}$.

Problem 12 (6 points)**.** Recall the definition of product forcing from Models of Set Theory I (e.g. in the lecture and Problem 28). Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing notions.

(a) Show that $\mathbb{P} \times \mathbb{Q}$ and $\mathbb{P} * \check{\mathbb{Q}}$ are forcing equivalent.

(b) Prove that if G is M-generic for some atomles forcing notion $\mathbb{P}$ then $G \times G$ is not M-generic for $\mathbb{P} \times \mathbb{P}$.

(c) Show that, in general, two-step iterations of partial orders are not antisymmetric. *For this reason, one generalizes forcing to preorders rather than partial orders.*

Problem 13 (6 points)**.** Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing notions.

(a) Prove that the following statements are equivalent:
   (1) $\mathbb{P} \times \mathbb{Q}$ is ccc.
   (2) $\mathbb{P}$ is ccc and $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}^{M}$ "$\check{\mathbb{Q}}$ is ccc".
   (3) $\mathbb{Q}$ is ccc and $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}}^{M}$ "$\check{\mathbb{P}}$ is ccc".

(b) Prove from MA_{ω_1} that $\mathbb{P} \times \mathbb{Q}$ is ccc if and only if $\mathbb{P}$ and $\mathbb{Q}$ are ccc.

Hint for (a): For "(1) $\to$ (2)" assume $p \Vdash_{\mathbb{P}}^{M}$ "$\dot{f} : \check{\omega}_1 \to \check{\mathbb{Q}}$ enumerates an antichain" and choose a suitable antichain in $\mathbb{P}$ below p. For the converse, consider the $\mathbb{P}$-name $\sigma = \{\langle \check{\xi}, p_\xi\rangle \mid \xi < \omega_1\}$ and show that whenever G is M-generic for $\mathbb{P}$, σ^G is countable.

Problem 14 (4 points)**.** Let T be a Suslin tree and $\mathbb{T} = \langle T, \supseteq \rangle$ be the corresponding forcing notion.

(a) Show that $\mathbb{T} \times \mathbb{T}$ is not ccc.

(b) Conclude from MA_{ω_1} that there are no Suslin trees.

Please hand in your solutions on Monday, 23.11.2015 before the lecture.