Models of Set Theory II - Winter 2015/2016

Prof. Dr. Peter Koepke, Regula Krapf Problem sheet 1

Problem 4 (4 points)**.** If $\mathbb{P}$ is a forcing notion and G is M-generic for $\mathbb{P}$, we say that $f \in ({}^\omega\omega)^{M[G]}$ is a *dominating real*, if for every $g \in ({}^\omega\omega)^M$ there is $n_0 \in \omega$ such that for all $n \geq n_0$, $g(n) < f(n)$. Let $[\omega]^\omega$ denote the set of infinite subsets of ω. For $x, y \in [\omega]^\omega$ we say that x *splits* y if both $y \cap x$ and $y \setminus x$ are in $[\omega]^\omega$. A set $x \in ([\omega]^\omega)^{M[G]}$ is said to be a *splitting real* if it splits every set in $([\omega]^\omega)^M$. Prove that if $M[G]$ contains a dominating real then it contains a splitting real.

Hint: If $f \in M[G]$ is dominating, one may assume that f is strictly increasing. Let $f^{n+1}(0) = f(f^n(0))$ and $f^0(0) = 0$ and onstruct a splitting real by taking the union of certain intervals of the form $[f^k(0), f^l(0))$ in a suitable way, where for $n, m \in \omega$, $[n, m) = \{k \in \omega \mid n \leq k < m)\}$.

Problem 5 (6 points)**.** Let $\mathbb{C}$ denote Cohen forcing. Prove the following statements:

(a) $\mathbb{C}$ adds splitting reals.

(b) $\mathbb{C}$ does not add dominating reals.

Hint for (b): Consider an enumeration $\langle p_n \mid n \in \omega \rangle$ of the conditions in $\mathbb{C}$ and $g(n) = \min\{k \in \omega \mid \exists p \leq_{\mathbb{C}} p_n (p \Vdash_{\mathbb{C}} \dot{f}(\check{n}) = \check{k})\}$ for a $\mathbb{C}$-name $\dot{f}$ for a real.

Problem 6 (6 points)**.** *Hechler forcing* $\mathbb{P}$ is the forcing notion whose conditions are of the form $p = \langle s_p, E_p \rangle$ such that $s_p : n \to \omega$ for some $n \in \omega$ and $E_p \subseteq {}^\omega\omega$ is a finite set of functions from ω to ω. The ordering is given by

$$p \leq_{\mathbb{P}} q \iff s_p \supseteq s_q \wedge E_p \supseteq E_q \wedge \forall f \in E_q \, \forall n \in \mathrm{dom}(s_p) \setminus \mathrm{dom}(s_q)(f(n) < s_p(n)).$$

(a) Show that $\mathbb{P}$ satisfies the c.c.c.

(b) Prove that $\mathbb{P}$ adds a dominating real.

(c) Prove that $\mathbb{P}$ adds Cohen reals, i.e. whenever G is M-generic for $\mathbb{P}$, then there is $H \in M[G]$ which is M-generic for $\mathrm{Fn}(\omega, 2, \aleph_0)$.

Please hand in your solutions on Monday, 09.11.2015 before the lecture.